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MULTI-LEVEL BANDWIDTH EFFICIENT BLOCK MODULATION CODES

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ABSTRACT

In this paper, we investigate the multi-level technique for combining block coding and modulation. The paper consists of four parts. In the first part, we present a formulation for signal sets on which modulation codes are to be constructed. Distance measures on a signal set are defined and their properties are developed. In the second part, we present a general formulation for multi-level modulation codes in terms of component codes with appropriate Euclidean distances. The distance properties, Euclidean weight distribution and linear structure of multi-level modulation codes are investigated. In the third part, several specific methods for constructing multi-level block modulation codes with interdependency among component codes are proposed. Given a multi-level block modulation code C with no interdependency among the binary component codes, the proposed methods give a multi-level block modulation code C' which has the same rate as C , a minimum squared Euclidean distance not less than that of code C , a trellis diagram with the same number of states as that of C and a smaller number of nearest neighbor codewords than that of C . In the last part, error performance of block modulation codes is analyzed for an AWGN channel based on soft-decision maximum likelihood decoding. Error probabilities of some specific codes are evaluated based on their Euclidean weight distributions and simulation results.

1. Introduction

One of the dramatic developments in bandwidth-efficient communications over the past few years is the introduction and rapid application of combined coding and bandwidth-efficient modulation, known as coded modulation, for reliable data transmission [1]. The basic concept of coded modulation is to encode information symbols onto an expanded channel signal set (relative to that needed for uncoded modulation). The channel signal set expansion provides the needed redundancy for error control without increasing bandwidth requirements, while coding is used to produce a certain interdependency between successive channel signals, such that only certain sequences of channel signals are permitted. Using properly designed coded modulation, significant coding gains over uncoded modulation schemes can be achieved without compromising bandwidth efficiency [1].

Based on code structure, there are two basic types of coded modulations: the trellis coded modulation (TCM) and the block coded modulation (BCM). TCM was first introduced by Ungerboeck in 1982 [1]. Since the publication of Ungerboeck's paper, there has been a great deal of research on the construction of TCM codes [2-16]. In this paper, we focus on BCM. Particularly, we investigate the powerful multi-level technique [17-26] for combining block coding and modulation. This multi-level technique allows us to construct bandwidth-efficient block modulation codes with arbitrary large minimum squared Euclidean distances from Hamming distance component codes (binary or nonbinary) in conjunction with proper signal mapping.

The presentation of this paper is organized as follows. In Section 2, we present a formulation for signal sets on which modulation codes are to be constructed. Each signal point is labeled by a string of symbols from a certain finite alphabet, say $\{0, 1\}$. Distance measures on a signal set are defined and their properties are developed. In Section 3, we provide a general formulation for multi-level modulation codes in terms of component codes over substrings of labeling symbols. Lower bounds on the minimum (squared Euclidean) distance of multi-level modulation codes are derived, and a sufficient condition under which the lower bounds give the exact minimum distance is given. In Section 4, linear multi-level modulation

codes are introduced and their weight structures are discussed. In Section 5, several specific methods for constructing multi-level block modulation codes are proposed. Most of the known block modulation codes are basic multi-level codes constructed from binary block component codes with no interdependency among them. One problem with the basic multi-level block modulation codes is the large number of nearest neighbor codewords (or path multiplicity) in comparison to TCM codes of the same complexity. To solve this problem, interdependency between consecutive levels of labeling of component codes must be taken into account. In Section 5, several methods for constructing modulation codes over two to four levels of a binary labeling are proposed. These proposed construction methods provide interdependency among component codes. Given a basic multi-level block modulation code C , the proposed methods give a nonbasic multi-level block modulation code C' which has the same rate as C , a minimum squared Euclidean distance not less than that of C , a trellis diagram with the same number of states as that of C and a smaller number of nearest neighbor codewords than that of C . In Section 6, error performance of block modulation codes is analyzed for an AWGN channel based on a soft-decision maximum likelihood decoding. Error probabilities of some multi-level block codes for 8-PSK, 16-PSK and 16-QASK modulations are evaluated based on their Euclidean weight distributions and simulation results. These codes are shown to provide significant coding gains over some uncoded reference modulation schemes with little or no bandwidth expansion. Most of these codes have simple trellis structure, and hence can be decoded with the soft-decision Viterbi decoding algorithm.

2. Signal Sets, Labeling and Distance Measures

In this section, we present a formulation for signal sets on which modulation codes are to be constructed. Each signal point is labeled by a string of symbols from a certain finite alphabet. Distance measures on a signal set are defined and their properties are developed.

Let S be a finite set on which a distance measure between two elements, s and s' , denoted $d(s, s')$, is defined and satisfies the following conditions:

$$d(s, s') = d(s', s) \geq 0, \quad (2.1)$$

$$d(s, s') = 0 \quad \text{if and only if} \quad s = s'. \quad (2.2)$$

This set S (or its product) represents a set of elementary signal points (e.g., an MPSK signal set), and the distance measure $d(s, s')$ denotes the distance measure between two signal points represented by s and s' respectively (e.g., the squared Euclidean distance between two signal points). The error performance of a modulation code over S is evaluated based on this distance measure.

For most signal sets, the following condition (S1) holds :

(S1) S is chosen as either an additive abelian group \bar{S} or a finite subset of \bar{S} , and the distance measure on S is the restriction to S of a distance measure d on \bar{S} such that for any two elements, s and s' , in \bar{S} ,

$$d(s, s') = d(s - s', 0), \quad (2.3)$$

where 0 denotes the zero element of the group \bar{S} and “-” denotes the inverse operation of the group addition.

Almost all the coded modulation techniques which have been studied so far are based on bits-to-signal point mapping through signal set partitioning introduced by Ungerboeck [1]. Many authors [1,11-16,18-21,25] have considered the problem of partitioning a signal constellation and labeling the parts (signal points) by strings of symbols from a certain finite alphabet, mostly the binary alphabet $\{0, 1\}$. The common point to all these labelings is that if two strings $a_1 a_2 \cdots a_\ell$ and $a'_1 a'_2 \cdots a'_\ell$ differ for the first time at the position i , then the corresponding signal points are at a distance at least d_i apart. In this paper, we also follow this idea. For a positive integer ℓ , we shall only consider a labeling whose set of label strings is of the following form:

$$L \triangleq \{a_1 a_2 \cdots a_\ell : a_i \in L_i \text{ for } 1 \leq i \leq \ell\}, \quad (2.4)$$

where L_i is a finite set of two or more symbols from the label alphabet for $1 \leq i \leq \ell$. Let λ denote the one-to-one mapping from L to S defined by a labeling. Hence each signal point in S is uniquely represented by a label string in L . The labeling L is said to have ℓ levels or length ℓ . For S and λ , the i -th distance parameter d_i of S with $1 \leq i \leq \ell$ is defined as follows [18]:

(1)

$$d_1 \triangleq \min \{d(s, s') : s, s' \in S \text{ and } s \neq s'\}. \quad (2.5)$$

(2) For $1 < i \leq \ell$,

$$\begin{aligned} d_i &\triangleq \min \{d(s, s') : s = \lambda(a_1 a_2 \cdots a_\ell), s' = \lambda(a'_1 a'_2 \cdots a'_\ell), \\ &\quad s \neq s', a_j \in L_j, a'_j \in L_j \text{ for } 1 \leq j \leq \ell, \\ &\quad \text{and } a_j = a'_j \text{ for } 1 \leq j < i\}. \end{aligned} \quad (2.6)$$

From the definition of d_i , we see that for $1 \leq i < \ell$,

$$d_i \leq d_{i+1}. \quad (2.7)$$

For a 2^ℓ -ary QASK signal set, S is chosen as a subset of 2^ℓ points from a 2-dimensional lattice. The distance $d(s, s')$ between two elements, s and s' , in S is chosen to be the squared Euclidean distance between signal points represented by s and s' respectively. A binary labeling L of length ℓ is chosen in such a way [1, 3] that for $1 < i \leq \ell$,

$$d_i = 2d_{i-1}. \quad (2.8)$$

Such a set S with the labeling L is denoted $S_{2^\ell, \text{QASK}}$.

For some signal sets satisfying condition (S1), the following condition (S2) also holds:

(S2) We can choose ℓ subsets, B_1, B_2, \dots, B_ℓ of \bar{S} which have the following properties:

- (1) For $1 \leq i \leq \ell$, $|B_i| \geq 2$, where $|X|$ denotes the number of elements in a set X .
- (2) S is the direct-sum of B_1, B_2, \dots, B_ℓ , denoted $B_1 + B_2 + \cdots + B_\ell$, i.e., for each element s in S , there are unique $b_i \in B_i$ for $1 \leq i \leq \ell$ such that

$$s = b_1 + b_2 + \cdots + b_\ell. \quad (2.9)$$

- (3) For $1 \leq i < \ell$, $b \in B_i$ and $b' \in B_i$,

$$d(b, b') = \min \{d(b + c, b' + c') : c \text{ and } c' \text{ in } B_{i+1} + B_{i+2} + \cdots + B_\ell\}. \quad (2.10)$$

For such a signal set, we will use the following labeling. For $1 \leq i \leq \ell$, choose a set L_i of $|B_i|$ symbols and a one-to-one mapping λ_i from L_i to B_i , and define the mapping λ as follows: For $a_i \in L_i$ with $1 \leq i \leq \ell$,

$$\lambda(a_1 a_2 \cdots a_\ell) \triangleq \lambda_1(a_1) + \lambda_2(a_2) + \cdots + \lambda_\ell(a_\ell). \quad (2.11)$$

A set S with the above labeling for which conditions (S1) and (S2) hold is said to be of direct-sum type. It follows from (2.3), (2.5), (2.6), (2.9) to (2.11) that the i -th distance parameter of S is

$$d_i = d[B_i], \quad (2.12)$$

where $d[X]$ denotes the minimum distance between different elements in a subset X of \bar{S} . The subsets, B_1, B_2, \dots, B_ℓ are said to form a composition of the set S .

For a 2^ℓ -ary PSK signal set, the integer group $\{0, 1, 2, \dots, 2^\ell - 1\}$ under the modulo- 2^ℓ addition is chosen as the set S (i.e., $S = \bar{S}$). Each element in S represents a point in the 2-dimensional 2^ℓ -ary PSK signal set. The distance measure $d(s, s')$ between the two elements, s and s' , in S is chosen to be the squared Euclidean distance between two signal points represented by s and s' respectively, and is given by

$$d(s, s') = 4 \sin^2 \left(2^{-\ell} \pi (s - s') \right). \quad (2.13)$$

For $1 \leq i \leq \ell$, we choose

$$B_i = \{0, 2^{i-1}\}. \quad (2.14)$$

Then S is the direct-sum of B_1, B_2, \dots, B_ℓ and the right-hand side of (2.9) is simply the standard binary representation of the integer s . In this case, B_1, B_2, \dots, B_ℓ are said to form a basic composition of S . It is easy to check that the condition of (2.10) holds. We use a binary labeling for S with $L_i = \{0, 1\}$, $\lambda_i(0) = 0$ and $\lambda_i(1) = 2^{i-1}$ for $1 \leq i \leq \ell$. Then each signal point in S is labeled by a sequence of ℓ binary digits. It follows from (2.12) to (2.14) that the i -th distance parameter of S is

$$d_i = 4 \sin^2 \left(2^{i-1-\ell} \pi \right), \quad (2.15)$$

for $1 \leq i \leq \ell$. From (2.15), we readily see that

$$d_\ell = 2d_{\ell-1}. \quad (2.16)$$

The above set S with distance measure given by (2.13) is denoted $S_{2^\ell, \text{PSK}}$. As another example of direct-sum type, let \bar{S} be the set of all integers which is an additive abelian group. For $1 \leq i \leq \ell$, let $B_i = \{-2^{i-1}, 2^{i-1}\}$. Define S as follows:

$$\begin{aligned} S &\triangleq \{b_1 + b_2 + \dots + b_\ell : b_i \in B_i \text{ for } 1 \leq i \leq \ell\} \\ &= \{-2^\ell + 1, -2^\ell + 3, \dots, -1, 1, \dots, 2^\ell - 3, 2^\ell - 1\} \\ &= B_1 + B_2 + \dots + B_\ell. \end{aligned}$$

For s and s' in \bar{S} , define the distance measure $d(s, s')$ on \bar{S} as $(s - s')^2$. Then it is easy to show that the property of (2.10) holds and $d_i = 2^{2i}$. The product, $S^2 = \{(s, s') : s, s' \in S\}$, may be used to represent a set of signal points for some modulation (a special case is used in [25]). The construction of a code of length n over the signal set S^2 is that of a code of length $2n$ over S .

Since the distance measure on $S_{2^l\text{-QASK}}$ or $S_{2^l\text{-PSK}}$ is not simple enough to be used effectively for constructing codes for 2^l-QASK or 2^l-PSK modulation, a simpler working distance measure is usually taken. If a multi-stage decoding algorithm is used for multi-level modulation codes, an appropriate working distance measure may be more useful than d itself. Such a working distance measure, denoted $g(\cdot, \cdot)$, is a real function on $S \times S$ which satisfies (2.1), (2.2) and the following condition: for s and s' in S ,

$$d(s, s') \geq g(s, s'). \quad (2.17)$$

Most modulation codes [1,3,15,17-19] are basically constructed based on the following distance measure $\underline{d}(\cdot, \cdot)$. For $s = \lambda(a_1 a_2 \dots a_l)$ and $s' = \lambda(a'_1 a'_2 \dots a'_l)$ in S , let $\underline{d}(s, s')$ be defined as follows:

(1) If $s = s'$, then

$$\underline{d}(s, s') \triangleq 0. \quad (2.18)$$

(2) Otherwise,

$$\underline{d}(s, s') \triangleq d_h, \quad (2.19)$$

where h denotes the first suffix such that $a_h \neq a'_h$.

It follows from the definitions of d_i and $\underline{d}(s, s')$ that for any two elements, s and s' , in S ,

$$d(s, s') \geq \underline{d}(s, s'). \quad (2.20)$$

Other examples of working distance measures are the Euclidean weight [1] (see Section 4) and the Hamming distance with proportionality in [20].

Let L be the label set for the signal set S . We define the distance measure between two strings in L as follows : For a distance measure g on S and two strings α and α' in L , let $g_L(\alpha, \alpha')$ be defined as

$$g_L(\alpha, \alpha') \triangleq g(\lambda(\alpha), \lambda(\alpha')). \quad (2.21)$$

Notation S will be used as a more generic notation than L . For a positive integer n , let X^n denote the set of all n -tuples over a set X . Let $g(\cdot, \cdot)$ be a measure, a real function, defined on X^2 (e.g. $d(\cdot, \cdot)$ or $\underline{d}(\cdot, \cdot)$). We extend the domain of g as follows: For two n -tuples, $\bar{v} = (v_1, v_2, \dots, v_n)$ and $\bar{v}' = (v'_1, v'_2, \dots, v'_n)$ over X ,

$$g(\bar{v}, \bar{v}') \triangleq \sum_{j=1}^n g(v_j, v'_j). \quad (2.22)$$

For a nonempty subset C of X^n , define the minimum distance of C with respect to measure $g(\cdot, \cdot)$, denoted $D[g, C]$, as follows:

$$D[g, C] \triangleq \min \{ g(\bar{v}, \bar{v}') : \bar{v}, \bar{v}' \in C \text{ and } \bar{v} \neq \bar{v}' \}. \quad (2.23)$$

(If $|C| = 1$, then $D[g, C]$ is defined as infinity.) For two real functions, $g(\cdot, \cdot)$ and $g'(\cdot, \cdot)$ defined on X^2 and a nonempty subset C of X^n , it follows from (2.22) and (2.23) that if $g(\cdot, \cdot) \geq g'(\cdot, \cdot)$, then

$$D[g, C] \geq D[g', C]. \quad (2.24)$$

We use $D[C]$ and $\underline{D}[C]$ to denote $D[d, C]$ and $D[\underline{d}, C]$ respectively for simplicity. It follows from (2.20) and (2.24) that

$$D[C] \geq \underline{D}[C]. \quad (2.25)$$

Let C be a block code of length n over S (or L) which represents either the 2^ℓ -PSK or the 2^ℓ -QASK signal set. If each component of codeword \bar{v} in C is mapped into the corresponding signal point in the 2-dimensional 2^ℓ -PSK or 2^ℓ -QASK signal set, we obtain a block 2^ℓ -PSK or 2^ℓ -QASK modulation code with minimum squared Euclidean distance $D[C]$. The effective rate of this code is given by [1],

$$R[C] = \frac{1}{2n} \log_2 |C|, \quad (2.26)$$

which is simply the average number of information bits transmitted by C per dimension.

3. Multi-level Block Modulation Code

The multi-level technique is a powerful method for constructing modulation codes with arbitrary large minimum squared Euclidean distance from component codes in conjunction with proper signal mapping. In this section, we present a general formulation for multi-level block

modulation codes in terms of component codes over substrings of labeling symbols. Lower bounds on the minimum distance of a multi-level code based on a distance measure g_L over a labeling L are derived.

Suppose a signal set S and a labeling L of ℓ levels for S are given. Since the mapping from L to S is one-to-one, constructing a code over S is equivalent to constructing a code over L . For constructing a general multi-level code over L , we must segment the labeling into sub-labeling and choose the starting symbol position of each sub-labeling. Let m be a positive integer not greater than ℓ , and let j_1, j_2, \dots, j_{m+1} be $m+1$ integers such that

$$1 = j_1 < j_2 < \dots < j_m < j_{m+1} = \ell + 1. \quad (3.1)$$

For $1 \leq i \leq m$, let $\ell^{(i)}$ be defined as

$$\ell^{(i)} \triangleq j_{i+1} - j_i,$$

and let $L^{(i)}$ denote the set of substrings from the j_i -th symbol to the $(j_{i+1} - 1)$ -th symbol of strings in L defined by (2.4), i.e.,

$$L^{(i)} \triangleq \{ a_{j_i} a_{j_i+1} \dots a_{j_{i+1}-1} : a_h \in L_h \text{ for } j_i \leq h < j_{i+1} \}. \quad (3.2)$$

Clearly,

$$L = L^{(1)} * L^{(2)} * \dots * L^{(m)},$$

where $*$ denotes the concatenation operation. For $1 \leq i \leq m$, $L^{(i)}$ is called the i -th level sub-labeling.

Consider an n -tuple $\bar{v} = (v_1, v_2, \dots, v_n)$ over L . For $1 \leq j \leq n$, the j -th component v_j of \bar{v} can be expressed as the following concatenation of substrings in $L^{(1)}$ to $L^{(m)}$:

$$v_j = v_{j1} v_{j2} \dots v_{jm}$$

where $v_{ji} \in L^{(i)}$ for $1 \leq i \leq m$. For $1 \leq i \leq m$, we form the following n -tuple over $L^{(i)}$:

$$\bar{v}^{(i)} = (v_{1i}, v_{2i}, \dots, v_{ni}). \quad (3.3)$$

This n -tuple $\bar{v}^{(i)}$ is called the i -th component n -tuple of \bar{v} , and \bar{v} is denoted as follows:

$$\bar{v} = \bar{v}^{(1)} * \bar{v}^{(2)} * \dots * \bar{v}^{(m)}. \quad (3.4)$$

For $1 \leq i \leq m$, let C_i be a block code of length n over $L^{(i)}$. From C_1, C_2, \dots, C_m , we form a block code of length n over L as follows:

$$C \triangleq \{ \bar{v}^{(1)} * \bar{v}^{(2)} * \dots * \bar{v}^{(m)} : \bar{v}^{(i)} \in C_i \text{ for } 1 \leq i \leq m \}. \quad (3.5)$$

Such a code is called a ℓ -level code with m components. We denote C with $C_1 * C_2 * \dots * C_m$ and C_i is called the i -th component code of C .

For a distance measure g_L on L and $1 \leq i \leq m$, let $g_L^{(i)}(w, w')$ with w and w' in $L^{(i)}$ be defined as follows:

$$\begin{aligned} g_L^{(i)}(w, w') \triangleq \min \{ & g_L(w_1 \dots w_{i-1} w w_{i+1} \dots w_m, w_1 \dots w_{i-1} w' w'_{i+1} \dots w'_m) : \\ & w_j \in L^{(j)} \text{ with } j = 1, \dots, i-1, i+1, \dots, m \text{ and} \\ & w'_j \in L^{(j)} \text{ with } j = i+1, \dots, m. \} \end{aligned} \quad (3.6)$$

For any real function $g_L(\cdot, \cdot)$ on $L \times L$, a lower bound on the minimum distance $D[g_L, C]$ of a multi-level code C based on the distance measure g_L is given in Lemma 1.

Lemma 1:

$$D[g_L, C] \geq \min_{1 \leq i \leq m} D[g_L^{(i)}, C_i], \quad (3.7)$$

where $g_L^{(i)}$ is defined by (3.6). The equality holds for $g_L = \underline{d}_L$ defined by (2.18), (2.19) and (2.21).

Proof: See Appendix A. △△

This lemma unifies the previous results [17, 18, 21, 25, 27, for $g = \underline{d}_L$, 20, for the Hamming distance with proportionality]. From (2.17) and (2.24), Lemma 1 gives the following lower bound on $D[C]$ ($= D[d, C]$),

$$D[C] \geq \min_{1 \leq i \leq m} D[g_L^{(i)}, C_i]. \quad (3.8)$$

Now we consider a special case for which $m = \ell$ and $g_L = \underline{d}_L$. C is formed from ℓ component codes. Let δ_i be the minimum Hamming distance of C_i . Then it follows from (2.18) to (2.20) and (3.6) that for two different symbols a and a' in L_i ,

$$\underline{d}_L^{(i)}(a_i, a'_i) = \delta_i \quad (3.9)$$

for $1 \leq i \leq \ell$. From (2.23) and (3.9), we have

$$D[d_L^{(i)}, C_i] = \delta_i d_i. \quad (3.10)$$

Combining (2.25), (3.8) and (3.10), we obtain the following lower bound on $D[C]$ [18, 27, 28]:

$$D[C] \geq \min_{1 \leq i \leq \ell} \delta_i d_i. \quad (3.11)$$

The above special case was first proposed by Imai and Hirakawa [17] and then by Ginzburg [18] and Sayegh [19]. A ℓ -level code with ℓ components is called a basic multi-level code. Most of the known block modulation codes are basic multi-level codes. A basic multi-level modulation code is constructed from ℓ Hamming distance component codes with no interdependency among them. Simple methods for constructing basic multi-level block codes for various types of modulations are given in [18,19,23-25,29].

For a signal set S of direct-sum type, we have stronger results on the minimum distance of a multi-level code which are given in Lemma 2.

Lemma 2: Suppose that S is of direct-sum type.

(1) If $\ell^{(i)} = 1$, then for a and a' in $L^{(i)} (= L_j)$,

$$d_L^{(i)}(a, a') = d(\lambda_j(a), \lambda_j(a')). \quad (3.12)$$

(2) For α and α' in $L^{(m)}$,

$$d_L^{(m)}(\alpha, \alpha') = d(\lambda^{(m)}(\alpha), \lambda^{(m)}(\alpha')), \quad (3.13)$$

where, for $a_{j_m} a_{j_m+1} \cdots a_\ell$ in $L^{(m)}$

$$\lambda^{(m)}(a_{j_m} a_{j_m+1} \cdots a_\ell) \triangleq \lambda_{j_m}(a_{j_m}) + \lambda_{j_m+1}(a_{j_m+1}) + \cdots + \lambda_\ell(a_\ell).$$

(3) If $\ell^{(i)} = 1$ for every i other than m , then

$$D[C] = \min_{1 \leq i \leq m} D[d_L^{(i)}, C_i], \quad (3.14)$$

$$D[w_L, C] = \min_{1 \leq i \leq m} D[w_L^{(i)}, C_i], \quad (3.15)$$

where w denotes the Euclidean weight defined by (4.4) in the next section.

(4) If $m = \ell$ and $|B_i| (= |L_i|) = 2$ for $1 \leq i \leq \ell$, then

$$D[C] = \min_{1 \leq i \leq \ell} \delta_i d_i, \quad (3.16)$$

where δ_i denotes the minimum Hamming distance of C_i .

Proof: See Appendix A. △△

In general, a symbol in L_i does not denote a single signal point but a family of subsets of signal points. For a signal set S of direct-sum type, a ℓ -level block modulation code C over S (instead of L) can be defined as the following sum of m component codes, C_1, C_2, \dots, C_m :

$$\begin{aligned} C &\triangleq C_1 + C_2 + \dots + C_m \\ &\triangleq \{ \bar{\mathbf{v}}^{(1)} + \bar{\mathbf{v}}^{(2)} + \dots + \bar{\mathbf{v}}^{(m)} : \bar{\mathbf{v}}^{(i)} \in C_i \text{ for } 1 \leq i \leq m \}, \end{aligned} \quad (3.17)$$

where the i -th component code C_i has symbols from $B_{j_i} + B_{j_i+1} + \dots + B_{j_{i+1}-1} \subseteq \bar{S}$ and “+” denotes the component wise “+” addition. If the assumption in (3) of Lemma 2 is satisfied, then

$$D[C_1 + C_2 + \dots + C_m] = \min_{1 \leq i \leq m} D[C_i]. \quad (3.18)$$

When a signal set S and a labeling L of length ℓ for S are given together, it is desirable for the labeling to display the detail distance structure of S as much as possible. However in the construction of multi-level codes, choosing the number of components, m , less than ℓ may results in a code with better performance than a basic ℓ -level code. This will be shown in Section 5.

For a block modulation code to be decoded with a soft-decision maximum likelihood decoding algorithm, it is desirable that the code has a trellis structure so that the Viterbi algorithm can be applied. A multi-level code has trellis structure if each of its component codes has trellis structure. A trellis diagram for the multi-level code C can be obtained by taking the direct product of trellis diagrams for its component codes. To reduce the decoding complexity, multi-level modulation codes can be decoded with a multi-stage decoding [17, 20, 25].

4. Linear Multi-level Codes

In this section we study multi-level codes with linear structure. Linear structure makes the error performance analysis of a code much easier.

Suppose the signal set S can be taken as an additive abelian group under addition $+$. For a distance measure g on S and an n -tuple $\bar{\mathbf{v}}$ over S , define $|\bar{\mathbf{v}}|_g$ as follows:

$$|\bar{\mathbf{v}}|_g \triangleq g(\bar{\mathbf{v}}, \bar{\mathbf{0}}), \quad (4.1)$$

where $\bar{0}$ denotes the all-zero n -tuple over S . The parameter $|\bar{v}|_g$ is called the weight of \bar{v} with respect to the measure g .

A block code C over S is said to be linear with respect to $+_a$, if C is closed under the component-wise addition $+_a$. Suppose that a distance measure (or a working distance measure) g on S satisfies the following condition: For s and s' in S ,

$$g(s, s') = g(s -_a s', 0), \quad (4.2)$$

where $-_a$ denotes the inverse of $+_a$ and 0 is the zero element in S with respect to $+_a$. Then it follows from (2.22), (2.23), (4.1) and (4.2) that for a linear block code C over S ,

$$D[g, C] = \min \{ |\bar{v}|_g : \bar{v} \in C - \{\bar{0}\} \}. \quad (4.3)$$

As a result, the error performance evaluation of C with respect to the distance measure g is reduced to that of C in terms of the weight measure $|\cdot|_g$.

For the sake of simplicity, we assume that the label L is the set of all binary strings of length ℓ . Then S can be taken as a binary vector space of dimension ℓ in the following sense. For two elements s and s' in S labeled with $a_1 a_2 \dots a_\ell$ and $a'_1 a'_2 \dots a'_\ell$ respectively, define $s \oplus s'$ as the element labeled with the binary string of length ℓ whose i -th symbol is the modulo-2 sum of a_i and a'_i for $1 \leq i \leq \ell$. Note that $d(s, s')$ is not necessarily equal to $d(s \oplus s', 0)$ (e.g., $S_{2\ell\text{-PSK}}$ with $\ell \geq 3$). However, the distance measure $\underline{d}(\cdot, \cdot)$ defined by (2.18) and (2.19) does satisfy the condition of (4.2) with respect to \oplus . The tightest measure satisfying (4.2) with respect to \oplus , denoted w , is given by

$$w(s, s') \triangleq \min \{ d(t, t') : t, t' \in S \text{ and } t \oplus t' = s \oplus s' \}. \quad (4.4)$$

$|\cdot|_w$ is the Euclidean weight measure first described in [1]. For example, consider $S_{8\text{-PSK}} = \{0, 1, \dots, 7\}$. For this case, $|s|_w = |s|_d$ for $s \neq 5$ and $|5|_w = d_3 - d_1 > |5|_d = d_1$, where d_1 , d_2 and d_3 are defined by (2.15).

Most of known block modulation codes are linear with respect to \oplus .

For $\bar{v} = (\lambda(v_{11}v_{12}\dots v_{1\ell}), \lambda(v_{21}v_{22}\dots v_{2\ell}), \dots, \lambda(v_{n1}v_{n2}\dots v_{n\ell})) \in S^n$, let $\beta(\bar{v})$ denote the following binary $n\ell$ -tuple:

$$\beta(\bar{v}) = (v_{11}, v_{12}, \dots, v_{1\ell}, v_{21}, v_{22}, \dots, v_{2\ell}, \dots, v_{n1}, v_{n2}, \dots, v_{n\ell}).$$

For a block code C of length n over S , we form the following binary block code $\beta[C]$ of length $n\ell$:

$$\beta[C] \triangleq \{ \beta(\bar{v}) : \bar{v} \in C \}. \quad (4.5)$$

If and only if $\beta[C]$ is linear, C is linear with respect to \oplus . For a linear block code C over S , the number of information bits, called the dimension, of C is defined to be that of $\beta[C]$. Let β^{-1} denote the inverse mapping of β . The dual code of C , denoted C^\perp , is defined as

$$C^\perp \triangleq \beta^{-1}[\beta[C]^\perp], \quad (4.6)$$

where $\beta[C]^\perp$ denotes the dual code [30] of $\beta[C]$.

If a code C over S is linear with respect to \oplus , then the complete weight distribution [31] of C over S is useful for evaluating the error performance of C . For an n -tuple $\bar{v} = (v_1, v_2, \dots, v_n)$ over S , the composition of \bar{v} , denoted $comp(\bar{v})$, is a 2^ℓ -tuple,

$$\bar{t} = (t_0, t_1, \dots, t_{2^\ell-1}),$$

where t_i is the number of components in \bar{v} equal to the binary string $a_1 a_2 \dots a_\ell$ such that $\sum_{j=1}^t a_j 2^{j-1} = i$. Let $N_C(\bar{t})$ denote the number of codewords \bar{v} in C with $comp(v) = \bar{t}$. Let T_{comp} be the set

$$T_{comp} = \{(t_0, t_1, \dots, t_{2^\ell-1}) : 0 \leq t_i \leq n \text{ with } 0 \leq i < 2^\ell\}.$$

Then

$$\{N_C(\bar{t}) : \bar{t} \in T_{comp}\}$$

is the complete weight distribution of C . For a distance measure g and a nonnegative real number δ , let $N_g(\delta)$ or $N_{g,C}(\delta)$ denote the number of codewords \bar{v} in C such that $|\bar{v}|_g = \delta$. $N_g(\delta)$ is used to evaluate the error performance of C (see Sec. 6). Once the complete weight distribution is known, $N_g(\delta)$ can be computed. If C is linear and $\log_2 |C|$ is moderate, then the complete weight distribution of C can be computed simply by generating all codewords of C . If $n\ell - \log_2 |C|$ is moderate, the complete weight distribution of C can be computed from that of the dual code C^\perp of C defined by (4.6) [31, 32].

5. Component Code Construction

In section 3, we presented a general formulation for multi-level modulation codes in terms of component codes over substrings of labeling symbols. Suppose a signal set S and its labeling L of length ℓ are given. To form a ℓ -level modulation code over S with m components, we must first construct the component codes with proper minimum distances based on a distance measure g_L . For $1 \leq i \leq m$, the i -th component is constructed based only on the i -th level sub-labeling $L^{(i)}$ of length $\ell^{(i)}$ and the distance measure $g_L^{(i)}$. Once the component codes are constructed, they are combined to form a ℓ -level modulation code by concatenating the m sub-labeling strings at each component position and then replacing each labeling string by its corresponding signal point in S . The simplest case is to construct basic ℓ -level modulation codes with $m = \ell$. One problem with basic multi-level block modulation codes over S is their large number of nearest neighbor codewords (path multiplicity), in comparison to trellis modulation codes, e.g., Ungerboeck codes [1], of the same complexity. To solve this problem, interrelation between consecutive levels of given labeling must be taken into account. In this section, we present several methods for constructing codes over two to four levels of a binary labeling with interdependency between consecutive levels.

For simplicity, we omit the superfix $^{(i)}$, write d_1, d_2, \dots for distance parameters $d_{j,i}$, $d_{j,i+1}, \dots$, define L as $\{0, 1\}^\ell$ and write d, \underline{d}, w for distance measures $d_L, \underline{d}_L, w_L$ respectively. We also define the following notations:

- (1) For $1 \leq i \leq n$, \bar{e}_i denotes the binary unit n -tuple whose i -th component is one and whose other components are zero.
- (2) P_n denotes the binary $(n, n-1)$ linear code which consists of all the even-weight binary n -tuples.
- (3) P_n^\perp denotes the dual code of P_n which consists of the all-zero and all-one n -tuples.
- (4) V_n denotes the vector space of all binary n -tuples.
- (5) $RM_{h,j}$ denotes the j -th order Reed-Muller code of length $n = 2^h$. A boolean polynomial $p(x_1, x_2, \dots, x_h)$ represents the binary 2^h -tuple whose i -th component is given by $p(i_1, i_2, \dots, i_h)$, where (i_1, i_2, \dots, i_h) is the standard binary representation of $i-1$ with the least significant bit i_1 .
- (6) For two j -tuples $\bar{u} = (u_1, u_2, \dots, u_j)$ and $\bar{v} = (v_1, v_2, \dots, v_j)$, let $\bar{u} \circ \bar{v}$ denote the $2j$ -tuple $(u_1, u_2, \dots, u_j, v_1, v_2, \dots, v_j)$. For a binary (n, k) linear code C with minimum Hamming

distance δ and an (n, k') linear subcode C' of C with minimum Hamming distance δ' , let $\mu(C, C')$ be defined as

$$\mu(C, C') \triangleq \{\bar{u} \circ (\bar{u} \oplus \bar{v}) : \bar{u} \in C \text{ and } \bar{v} \in C'\}. \quad (5.1)$$

Then $\mu(C, C')$ is a $(2n, k + k')$ linear code with minimum Hamming distance $\min\{2\delta, \delta'\}$. This μ -construction is a special case of the $|u|u + v|$ construction [31, p.76] in that C' is restricted to be a subcode of C . It is known [31] that for $0 \leq j \leq h$,

$$RM_{h,j} = \mu(RM_{h-1,j}, RM_{h-1,j-1}), \quad \text{where } RM_{h-1,-1} = \{\bar{0}\}. \quad (5.2)$$

5.1 Code Construction for the Case with $\ell = 2$ and $g = d$

A. Gray Code Indexing Method

This construction method [15, 20] has been proposed for the special case for which the following condition holds:

$$d_2 = 2d_1. \quad (5.3)$$

For a binary $2n$ -tuple $\bar{v} = (v_1, v_2, \dots, v_{2n})$, define two binary n -tuples, $\varphi_1(\bar{v}) = (u_1, u_2, \dots, u_n)$ and $\varphi_2(\bar{v}) = (u'_1, u'_2, \dots, u'_n)$ as follows: For $1 \leq j \leq n$,

$$(1) \text{ If } v_j = v_{j+n} = 0, \text{ then } u_j = u'_j = 0; \quad (5.4)$$

$$(2) \text{ If } v_j = 0 \text{ and } v_{j+n} = 1, \text{ then } u_j = 1 \text{ and } u'_j = 0; \quad (5.5)$$

$$(3) \text{ If } v_j = 1 \text{ and } v_{j+n} = 0, \text{ then } u_j = u'_j = 1; \text{ and} \quad (5.6)$$

$$(4) \text{ If } v_j = v_{j+n} = 1, \text{ then } u_j = 0 \text{ and } u'_j = 1. \quad (5.7)$$

Let $\varphi(\bar{v})$ denote the following n -tuple over $L = \{0, 1\}^2$:

$$\varphi(\bar{v}) = \varphi_1(\bar{v}) * \varphi_2(\bar{v}). \quad (5.8)$$

For two binary $2n$ -tuples, \bar{v} and \bar{v}' , it follows from the condition of (5.3) and definition of the mapping φ that

$$d(\varphi(\bar{v}), \varphi(\bar{v}')) = |\bar{v} \oplus \bar{v}'|_H \cdot d_1, \quad (5.9)$$

where $|\bar{v}|_H$ denote the Hamming weight of \bar{v} and \oplus denotes the component-wise modulo-2 addition. Let C_b be a binary code of length $2n$ and minimum Hamming distance δ . Define the following block code of length n over $L = \{0, 1\}^2$,

$$\varphi[C_b] = \{\varphi(\bar{v}) : \bar{v} \in C_b\}. \quad (5.10)$$

Clearly, $\varphi[C_b]$ is a two-level code with two levels of interrelated labeling. If C_b is linear, we can readily see that $\varphi[C_b]$ is also linear with respect to \oplus . It follows from (5.9) and (5.10) that

$$|\varphi[C_b]| = |C_b|, \quad (5.11)$$

$$D[\varphi[C_b]] = \delta d_1. \quad (5.12)$$

This construction will be used as a part of the construction presented in Section 5.2.

It follows from (5.4) to (5.7) that

$$\varphi(\bar{u} \circ (\bar{u} \oplus \bar{v})) = \bar{v} * \bar{u}. \quad (5.13)$$

Hence,

$$\varphi[\mu(C_1, C_2)] = C_2 * C_1. \quad (5.14)$$

From (5.14) we see that to derive a nonbasic two-level code with the Gray code indexing method, we need to choose C_b which cannot be constructed by the $|u|u + v|$ construction.

B. Cross-Over Construction

Now we consider the case where the distance condition of (5.3) does not hold. Let C_{b1} be a binary (n, k_1) linear code with minimum Hamming distance δ_1 . C_{b1} may consist of only the all-zero n -tuple. In such a case, δ_1 is defined to be infinite. Let C_{b2} be a binary (n, k_2) linear code. Let f be a linear mapping from C_{b2} to the set

$$(V_n - C_{b1}) \cup \{\bar{0}\}.$$

For \bar{u} and \bar{v} in C_{b2} , $f(\bar{u} \oplus \bar{v}) = f(\bar{u}) \oplus f(\bar{v})$. Now we define a block code of length n over $L (= \{0, 1\}^2)$ as follows:

$$F(C_{b1}, f, C_{b2}) \triangleq \{(\bar{u} \oplus f(\bar{v})) * \bar{v} : \bar{u} \in C_{b1} \text{ and } \bar{v} \in C_{b2}\}. \quad (5.15)$$

It is clear that $F(C_{b1}, f, C_{b2})$ contains $k_1 + k_2$ information bits, and if $f(\bar{v}) = \bar{0}$ for every \bar{v} in C_{b2} , $F(C_{b1}, f, C_{b2})$ is simply the basic two-level code $C_{b1} * C_{b2}$.

Next we examine the minimum distance of $F(C_{b1}, f, C_{b2})$. Define the following linear subcode of C_{b2} :

$$C_{b2,0} \triangleq \{\bar{v} \in C_{b2} : f(\bar{v}) = \bar{0}\}. \quad (5.16)$$

Let $\delta_{2,0}$ denote the minimum Hamming distance of $C_{b2,0}$. Define a subcode of $F(C_{b1}, f, C_{b2})$ as follows:

$$C' \triangleq \{(\bar{u} \oplus f(\bar{v})) * \bar{v} : \bar{u} \in C_{b1} \text{ and } \bar{v} \in C_{b2} - C_{b2,0}\}. \quad (5.17)$$

Since $F(C_{b1}, f, C_{b2}) - C' = C_{b1} * C_{b2,0}$, it follows from Lemma 1 and (3.10) that

$$D[F(C_{b1}, f, C_{b2})] = \min\{\delta_1 d_1, \delta_{2,0} d_2, D[C']\}. \quad (5.18)$$

Both $\delta_{2,0}$ and $D[C']$ depend on the mapping f . Define the following set of n -tuples in $(\{0, 1\}^n - C_{b1}) \cup \{\bar{0}\}$:

$$f[C_{b2}] = \{f(\bar{v}) : \bar{v} \in C_{b2}\}. \quad (5.19)$$

Let $\delta_H[X]$ denote the minimum nonzero Hamming weight of a set X of binary tuples of the same length. Let $(\bar{u} \oplus f(\bar{v})) * \bar{v}$ be an n -tuple in C' . Since $\bar{v} \notin C_{b2,0}$, $f(\bar{v}) \neq \bar{0}$. Since $\bar{u} \in C_{b1}$ and $C_{b1} \cap f[C_{b2}] = \{\bar{0}\}$, $\bar{u} \oplus f(\bar{v}) \neq \bar{0}$. This implies the following inequality:

$$D[C'] \geq \delta_H[C_{b1} \oplus f[C_{b2}]] d_1, \quad (5.20)$$

where for subsets X and Y of $\{0, 1\}^n$,

$$X \oplus Y \triangleq \{\bar{u} \oplus \bar{v} : \bar{u} \in X \text{ and } \bar{v} \in Y\}. \quad (5.21)$$

Since $\delta_1 \geq \delta_H[C_{b1} \oplus f[C_{b2}]]$, it follows from (5.18) and (5.20) that we obtain the following lower bound on $D[F(C_{b1}, f, C_{b2})]$:

$$D[F(C_{b1}, f, C_{b2})] \geq \min\{\delta_{2,0} d_2, \delta_H[C_{b1} \oplus f[C_{b2}]] d_1\}. \quad (5.22)$$

Recently, Tanner [20, Theorem 3] has given a code construction method in terms of a parity-check matrix which actually corresponds to a special case of the above construction with:

(1) $d_1 = 0.586$ and $d_2 = 2$,

- (2) C_{b2} is V_n ,
- (3) $f(\bar{v}) = \bar{0}$ for any even-weight n -tuple \bar{v} and $f(\bar{v})$ is a fixed n -tuple \bar{u}_0 for any odd-weight n -tuple \bar{v} , and
- (4) \bar{u}_0 and C_{b1} generate a code with minimum Hamming distance 7.

In this case, $C_{b2,0}$ is the set of all the even-weight n -tuple with $\delta_{2,0} = 2$. From condition (4), $\delta_H[C_{b1} \oplus f[C_{b2}]] = 7$. Consequently, the right side of (5.22) is 4, i.e.,

$$D[F(C_{b1}, f, C_{b2})] \geq 4.$$

In the following we present two specific cross-over constructions. Codes constructed based on these methods have smaller numbers of nearest neighbor codewords than that of their corresponding basic two-level codes.

B.1 Class-1 Cross-over Construction

Let C_i with $i = 1$ or 2 be a binary (n, k_i) linear code with minimum Hamming weight δ_i . Consider the basic 2-level code $C_1 * C_2$, denoted C . Let $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{k_1}\}$ be a basis of C_1 , and let r_2 denote $n - k_2$. Suppose that $k_1 \geq r_2$, $|\bar{u}_{r_2}|_H = \delta_1$ and the last component of \bar{u}_{r_2} is zero. Let H be a parity-check matrix of C_2 whose last column is the transposition of $(0, 0, \dots, 0, 1)$. For $1 \leq i \leq r_2$, let \bar{h}_i be the i -th row of H . Now we define $F(C_{b1}, f, C_{b2})$, denoted C' , such that $|C'| = |C|$ and $D[C'] \geq D[C]$.

- (1) Let C_{b1} be the $(n, k_1 - r_2)$ linear subcode of C_1 generated by $\bar{u}_{r_2+1}, \bar{u}_{r_2+2}, \dots, \bar{u}_{k_1}$.
- (2) Define C_{b2} as V_n .
- (3) For binary n -tuple \bar{v} , define $f(\bar{v})$ as follows:

$$f(\bar{v}) \triangleq \sum_{i=1}^{r_2} (\bar{h}_i, \bar{v}) \bar{u}_i, \quad (5.23)$$

where (\bar{h}_i, \bar{v}) is the inner product of \bar{h}_i and \bar{v} .

Then, $f(\bar{v}) = \bar{0}$ if and only if $\bar{v} \in C_2$, that is, $C_{b2,0} = C_2$ and $\delta_{2,0} = \delta_2$. Since $C_{b1} \oplus f[C_{b2}] = C_1$, it follows from (5.22) that

$$D[C'] \geq \min\{\delta_1 d_1, \delta_2 d_2\}, \quad (5.24)$$

where the equality holds if $\delta_1 d_1 \geq \delta_2 d_2$. Note that $|f(\bar{e}_n) * \bar{e}_n|_d = |\bar{u}_{r_2} * \bar{e}_n|_d \geq \delta_1 d_1 + d_2$. If $\delta_1 d_1 \leq \delta_2 d_2$, then

$$N_{dC'}(\delta_1 d_1) < N_{dC}(\delta_1 d_1). \quad (5.25)$$

B.2 Class-2 Cross-over Construction

Next we consider the case where $\delta_1 d_1 > \delta_2 d_2$. We restrict C_1 and C_2 to be constructed by the μ -construction defined by (5.1).

For an even positive integer n , let C_{i1} be a binary $(n/2, k_{i1})$ linear code with minimum Hamming distance $\delta_i/2$ or greater, and let C_{i2} be an $(n/2, k_{i2})$ linear subcode of C_{i1} with minimum Hamming distance δ_i . Then $\mu(C_{i1}, C_{i2})$, denoted C_i , is an $(n, k_{i1} + k_{i2})$ linear code with minimum Hamming distance δ_i .

Suppose that $\{\bar{u}_{ij} : 1 \leq j \leq k_{i1}\}$ is a basis of C_{i1} and $\{\bar{u}_{ij} : k_{i1} - k_{i2} < j \leq k_{i1}\}$ is a basis of C_{i2} . Then the following set of binary n -tuples is a basis of C_i :

$$\{\bar{u}_{ij} \circ \bar{u}_{ij} : 1 \leq j \leq k_{i1} - k_{i2}\} \cup \{\bar{u}_{ij} \circ \bar{0}, \bar{0} \circ \bar{u}_{ij} : k_{i1} - k_{i2} < j \leq k_{i1}\}. \quad (5.26)$$

It is easy to show that the dual code of C_i , denoted C_i^\perp , is given by

$$C_i^\perp \triangleq \mu(C_{i2}^\perp, C_{i1}^\perp). \quad (5.27)$$

Now we define $F(C_{b1}, f, C_{b2})$, denoted $\gamma(C_1, C_2)$, such that $|\gamma(C_1, C_2)| = |C_1 * C_2| = 2^{k_{11} + k_{12} + k_{21} + k_{22}}$ and $\underline{D}[\gamma(C_1, C_2)] \geq \min\{\delta_1 d_1, \delta_2 d_2\} = \underline{D}[C_1 * C_2]$. For simplicity, assume that

$$k_{11} - k_{12} \geq k_{21} - k_{22}. \quad (5.28)$$

Define C_{b1} , C_{b2} and f as follows:

(1)

$$C_{b1} \triangleq \mu(C'_{11}, C_{12}), \quad (5.29)$$

where C'_{11} is the linear subcode of C_{11} generated by $\{\bar{u}_{1j} : k_{21} - k_{22} < j \leq k_{11}\}$.

(2)

$$C_{b2} \triangleq C_{21} \circ C_{21}, \quad (5.30)$$

where for sets X and Y of n -tuples, $X \circ Y \triangleq \{\bar{u} \circ \bar{v} : \bar{u} \in X, \bar{v} \in Y\}$.

(3) Let $\bar{h}_1, \bar{h}_2, \dots, \bar{h}_{k_{21} - k_{22}}$ be linearly independent $n/2$ -tuples in $C_{22}^\perp - C_{21}^\perp$. For binary $n/2$ -tuples \bar{v} and \bar{v}' in C_{21} ,

$$f(\bar{v} \circ \bar{v}') \triangleq \left(\sum_{i=1}^{k_{21} - k_{22}} (\bar{h}_i, \bar{v}') \bar{u}_{1i} \right) \circ \left(\sum_{i=1}^{k_{21} - k_{22}} (\bar{h}_i, \bar{v}) \bar{u}_{1i} \right). \quad (5.31)$$

Note that $f(\bar{v} \circ \bar{v}') = \bar{0}$ if and only if both \bar{v} and \bar{v}' are in C_{22} . The key of this construction is "crossing" in that f maps the left half (or the right half) of the second component binary n -tuple into the right half (or the left half) of the first component binary n -tuple. The distance and weight properties are characterized by Lemma 3.

Lemma 3:

(1)

$$D[\gamma(C_1, C_2)] \geq \min\{\delta_1 d_1, \delta_2 d_2\}. \quad (5.32)$$

(2) If $\delta_1 d_1 > \delta_2 d_2$ and $k_{21} > k_{22}$, then the equality holds in (5.32) and

$$\begin{aligned} N_{\gamma(C_1, C_2)}(\delta_2 d_2) &= 2N_{H, C_{22}}(\delta_2) \\ &< N_{H, C_2}(\delta_2) = N_{\gamma(C_1 * C_2)}(\delta_2 d_2). \end{aligned} \quad (5.33)$$

where $N_{H, C}(\delta)$ denotes the number of codewords of Hamming weight δ in code C .

(3) If $C_1 * C_2$ has a t -section trellis diagram with s states, then $\gamma(C_1, C_2)$ has a t -section trellis diagram with s states.

Proof: See Appendix B. $\Delta\Delta$

Example 5.1: Assume that $4d_1 > d_2$ (e.g., $S_8\text{-PSK}$). Let p and q be nonnegative integers such that $q + 2 < p$, and let C_1 and C_2 be defined as

$$\begin{aligned} C_1 &\triangleq RM_{p,q} = \mu(RM_{p-1,q}, RM_{p-1,q-1}), \\ C_2 &\triangleq RM_{p,q+2} = \mu(RM_{p-1,q+2}, RM_{p-1,q+1}). \end{aligned}$$

Then $n = 2^p$, $\delta_1 = 2^{p-q}$, $k_{11} = \sum_{j=0}^q \binom{p-1}{j}$, $k_{12} = \sum_{j=0}^{q-1} \binom{p-1}{j}$, $\delta_2 = 2^{p-q-2}$, $k_{21} = \sum_{j=0}^{q+2} \binom{p-1}{j}$, $k_{22} = \sum_{j=0}^{q+1} \binom{p-1}{j}$, $k_{11} - k_{12} = \binom{p-1}{q}$, and $k_{21} - k_{22} = \binom{p-1}{q+2}$. Suppose that $p \leq 2q + 3$. Then $k_{11} - k_{12} \geq k_{21} - k_{22}$, and $D[\gamma(C_1, C_2)] = 2^{p-q-2} d_2$. This gives a class of two-level codes.

(1) As an example, consider the case where $p = 3$ and $q = 0$. Then $n = 8$, $C_1 = P_8^\perp = \mu(P_4^\perp, \{\bar{0}\})$, $k_{11} = 1$, $k_{12} = 0$, $C_2 = P_8 = \mu(V_4, P_4)$, $k_{21} = 4$, $k_{22} = 3$, $C_{b1} = \{\bar{0}\}$, $C_{b2} = V_4 \circ V_4 = V_8$ and mapping f is defined as follows:

$$f(\bar{e}_i) \triangleq (0, 0, 0, 0, 1, 1, 1, 1), \quad \text{for } 1 \leq i \leq 4,$$

$$f(\bar{e}_i) \triangleq (1, 1, 1, 1, 0, 0, 0, 0), \quad \text{for } 5 \leq i \leq 8.$$

Then the code,

$$\gamma(C_1, C_2) \triangleq F(C_{b1}, f, C_{b2}) = \{f(\bar{v}) * \bar{v} : \bar{v} \in V_8\},$$

has 8 information bits, and minimum weight $2d_2$ with respect to the distance measure \underline{d} . This code has an 8-section trellis diagram with 4 states as shown in Figure 1.

(2) Next consider the case where $p = 4$ and $q = 1$. Then, $n = 16$, $C_1 = RM_{4,1} = \mu(RM_{3,1}, P_8^\perp)$, $k_{11} = 4$, $k_{12} = 1$, $C_2 = P_{16} = \mu(V_8, P_8)$, $k_{21} = 8$, $k_{22} = 7$, C_{b1} can be chosen to be the $(16, 4)$ linear code generated by four binary 16-tuples represented by boolean polynomials $1, x_2, x_3$, and x_4 respectively, and $C_{b2} = V_8 * V_8 = V_{16}$. The mapping f from C_{b2} to $RM_{4,2}$ is defined as follows:

$$f(\bar{e}_i) \triangleq x_1(x_4 \oplus i_4), \quad \text{for } 1 \leq i \leq 16, \quad (5.34)$$

where $i_4 = 0$ for $1 \leq i \leq 8$ and $i_4 = 1$ for $9 \leq i \leq 16$.

(3) For four combinations of p and q , $N_{d,C}(\underline{D}[C])$ for $\gamma(C_1, C_2)$ is compared with that for $C_1 * C_2$ in Table-1. The number of states of a trellis diagram for $C_1 * C_2$ is computed based on the number of states of a trellis diagram for $RM_{p,q}$ [33].

(4) Consider the case where the signal set is $S_{8\text{-PSK}}$. Let $C^{(1)}$ and $C^{(2)}$ denote 8-PSK codes, $\gamma(P_8^\perp, P_8) * V_8$ and $\gamma(RM_{4,1}, P_{16}) * V_{16}$, respectively. Now, we compare these two codes with two corresponding 8-PSK codes, $C_B^{(1)} \triangleq P_8^\perp * P_8 * V_8$ and $C_B^{(2)} \triangleq RM_{4,1} * P_{16} * V_{16}$. We find that

$$R[C^{(1)}] = R[C_B^{(1)}] = 1, \quad R[C^{(2)}] = R[C_B^{(2)}] = 9/8,$$

$$D[C^{(1)}] = D[C_B^{(1)}] = D[C^{(2)}] = D[C_B^{(2)}] = 4,$$

$$N_{d,C^{(1)}}(4) = 56, \quad N_{d,C_B^{(1)}}(4) = 120,$$

$$N_{d,C^{(2)}}(4) = 240, \quad N_{d,C_B^{(2)}}(4) = 496.$$

We see that $C^{(1)}$ and $C^{(2)}$ have smaller path multiplicities than those of $C_B^{(1)}$ and $C_B^{(2)}$. Both $C^{(1)}$ and $C_B^{(1)}$ have 8-section trellis diagrams with 4 states, and both $C^{(2)}$ and $C_B^{(2)}$ have 4-section trellis diagrams with 16 states. These four codes are linear with respect to modulo-8 addition and $C_B^{(1)}$, $C^{(2)}$ and $C_B^{(2)}$ are invariant under 45° phase shift and $C^{(1)}$ is invariant under 90° phase shift [35]. Performance analysis for these codes is shown in the next section. $\Delta\Delta$

It follows from (5.26) and (5.29) that

$$C_{b1} = C'_{b1} \oplus (C_{12} \circ C_{12}),$$

where C'_{b1} is a binary $(n, k_{11} - k_{12} - k_{21} + k_{21})$ linear code generated by $\{\bar{u}_{1j} \circ \bar{u}_{1j} : k_{21} - k_{22} < j \leq k_{11} - k_{12}\}$. If C_{12} and C_{21} are constructed by the μ -construction, the class-2 cross-over construction can be recursively applied to $C_{12} * C_{21}$.

If inequality (5.28) is not true, then modify the construction as follows:

$$(1) \quad C_{b1} \triangleq C_{12} \circ C_{12},$$

$$(2) \quad C_{b2} \triangleq \mu(C'_{21}, C_{22}),$$

where C'_{21} is the linear subcode of C_{21} generated by $\{\bar{u}_{2j} : k_{11} - k_{12} < j \leq k_{21}\}$.

(3) In (5.31), replace $k_{21} - k_{22}$ by $k_{11} - k_{12}$.

5.2 Code Construction for the Case with $\ell \geq 3$, $g = \underline{d}$ or w and $d_\ell = 2d_{\ell-1}$

The two constructions stated in (A) and (B) of Section 5.1 can be combined for the case where $\ell = 3$ and the following condition holds:

$$d_3 = 2d_2. \quad (5.35)$$

Let C_{b1} be a binary (n, k_1) linear code with minimum Hamming distance δ_1 . Let C_{b2} be a binary $(2n, k_2)$ linear code. Let f be a linear mapping from C_{b2} to the set

$$(V_n - C_{b1}) \cup \{\bar{0}\}.$$

Now we define a block code of length n over the set $L(= \{0, 1\}^3)$ as follows:

$$H(C_{b1}, f, C_{b2}) \triangleq \{(\bar{u} \oplus f(\bar{v})) * \varphi_1(\bar{v}) * \varphi_2(\bar{v}) : \bar{u} \in C_{b1} \text{ and } \bar{v} \in C_{b2}\} \quad (5.36)$$

where the mappings φ_1 and φ_2 are defined by (5.4) to (5.7). It is clear that

$$|H(C_{b1}, f, C_{b2})| = 2^{k_1 + k_2}. \quad (5.37)$$

It can be shown that

$$\begin{aligned} D[H(C_{b1}, f, C_{b2})] &= \min\{\delta_1 d_1, \delta_{2,0} d_2, D[C]\} \\ &\geq \min\{\delta_{2,0} d_2, \delta_H[C_{b1} \oplus f[C_{b2}]] d_1\}, \end{aligned} \quad (5.38)$$

where $\delta_{2,0}$ denotes the minimum Hamming distance of $C_{b2,0} (\triangleq \{\bar{v} \in C_{b2} : f(\bar{v}) = \bar{0}\})$ and $C' \triangleq \{(\bar{u} \oplus f(\bar{v})) * \varphi_1(\bar{v}) * \varphi_2(\bar{v}) : \bar{u} \in C_{b1} \text{ and } \bar{v} \in C_{b2} - C_{b2,0}\}$. The class-1 or 2 cross-over construction can be applied to this case.

We use an example to illustrate the above construction method and give a class of zero-tail 4-state Ungerboeck's TCM codes.

Example 5.2: Let \bar{g}_1 and \bar{g}_2 be two binary $2n$ -tuples defined as follows:

$$\begin{aligned}\bar{g}_1 &\triangleq (1, 0, 1, 0, \dots, 0), \\ \bar{g}_2 &\triangleq (\underbrace{1, 0, \dots, 0}_n, 1, 0, \dots, 0).\end{aligned}$$

Let C_{b1} be the trivial code $\{\bar{0}\}$. Then $\delta_1 = \infty$. Let C_{b2} be the binary $(2n, 2n - 2)$ linear code generated by the set,

$$\{\sigma^i \bar{g}_1 : 0 \leq i < n - 2\} \cup \{\sigma^i \bar{g}_2 : 0 \leq i < n\},$$

where $\sigma^j \bar{v}$ denotes the tuple obtained from \bar{v} by cyclically shifting \bar{v} to the right j places. Then $\varphi_1[C_{b2}]$ is the binary $(n, n - 2)$ linear code generated by the set,

$$\{\sigma^i(1, 0, 1, 0, 0, \dots, 0) : 0 \leq i < n - 2\},$$

and $\varphi_2[C_{b2}]$ is simply the vector space, $\{0, 1\}^n$. Let f be the mapping from C_{b2} to $\{0, 1\}^n$ such that

$$\begin{aligned}f(\sigma^i \bar{g}_1) &\triangleq \sigma^i(0, 1, 0, \dots, 0) \text{ for } 0 \leq i < n - 2, \\ f(\sigma^i \bar{g}_2) &\triangleq \bar{0} \text{ for } 0 \leq i < n.\end{aligned}$$

Then, it follows from (5.36) that $H(C_{b1}, f, C_{b2})$, denoted C , is a linear code of length n over $L(= \{0, 1\}^3)$ with effective rate $R[C] = (n - 1)/n$.

Now we continue to determine $D[C]$. Note that $C_{b2,0}$ is the binary $(2n, n)$ linear code generated by the set,

$$\{\sigma^i \bar{g}_2 : 1 \leq i < n\}.$$

For any $2n$ -tuple $\bar{v} \in C_{b2} - C_{b2,0}$, \bar{v} can be expressed as

$$\bar{v} = \sigma^{i_1} \bar{g}_1 + \sigma^{i_2} \bar{g}_1 + \dots + \sigma^{i_h} \bar{g}_1 + \bar{u}$$

where $1 \leq h \leq n-2$, $0 \leq i_1 < i_2 < \dots < i_h < n-2$ and $\bar{u} \in C_{b2,0}$. Then the $(i_1 + 1)$ -th and $(i_h + 3)$ -th components of $\varphi_1(\bar{v})$ are equal to 1. On the other hand, the $(i_1 + 1)$ -th and $(i_h + 3)$ -th components of $f(\bar{v})$ are equal to zero. Consider the weight composition, $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, of the n -tuple $\bar{w} = f(\bar{v}) * \varphi_1(\bar{v}) * \varphi_2(\bar{v})$. Then the following inequality,

$$t_2 + t_6 \geq 2$$

holds. Since $t_1 + t_3 + t_5 + t_7 \geq 1$ and

$$|\bar{w}|_{\underline{d}} = (t_1 + t_3 + t_5 + t_7)d_1 + (t_2 + t_6)d_2 + t_4d_3,$$

we have that

$$D[C'] \geq d_1 + 2d_2.$$

Since $\delta_1 = \infty$ and $\delta_{2,0} = 2$, it follows from (5.38) that

$$D[C] = 2d_2.$$

Consider the case for which the signal set is $S_{8\text{-PSK}}$. Then C is a zero-tail 4-state Ungerboeck's 8-PSK TCM code[1] with minimum squared Euclidean distance $D[C] = 4(d_2 = 2)$. The number of codewords with minimum squared Euclidean distance from the all-zero codeword $\bar{0}$ is n . This code C is invariant only under 180° phase rotation [35]. Similarly, zero-tail Ungerboeck's TCM codes with 8 or more states can be constructed [24, 34]. $\Delta\Delta$

For the last three levels of S of direct-sum type (e.g., $S_{2^\ell\text{-PSK}}$ with $\ell \geq 3$), lower bounds on the minimum Euclidean weight w tighter than those on \underline{d} can be used.

Example 5.3: Suppose we want to construct an 8-PSK code C of length $n = 16$ with $R[C] = 1$ and $D[C] = d_2 + 4d_1$. Let $n = 16$, $C_{b1} = \{\bar{0}\}$ and $C_{b2} = \{0, 1\}^{32}$. For $1 \leq i \leq 32$, let \bar{e}_i be the binary unit 32-tuple whose i -th component is one and whose other components are zero. Define the mapping f from $C_{b2} = \{0, 1\}^{32}$ to $RM_{4,2}$ as follows:

$$f(\bar{e}_1) \triangleq x_3(1 \oplus x_4), \quad (5.39)$$

$$f(\bar{e}_i) \triangleq i_3 \oplus \sum_{j=1}^4 i_j x_j, \quad \text{for } 1 < i \leq 16 \text{ or } 17 < i \leq 32, \quad (5.40)$$

$$f(\bar{e}_{17}) \triangleq 1, \quad (5.41)$$

where $(i_1, i_2, i_3, i_4, i_5)$ is the standard binary representation of $i - 1$. It can be shown that (1) $D[C] = D[w, C] = d_2 + 4d_1 = 4.344$, (2) $N_{d,c}(d_2 + 4d_1) = 36$, and (3) this code has a 4-section trellis diagram with 2^8 states. $\Delta\Delta$

For the case where $\ell = 4$, $d_2 = 2d_1$ and $d_4 = 2d_3$, the Gray code indexing method can be used for the first and second levels as well as the third and fourth levels. The cross-over construction also can be applied to this case. For $i = 1$ or 2 , let C_{b_i} be a binary $(2n, k_i)$ linear code with minimum Hamming distance δ_i . Let f be the mapping from C_{b_2} to the set $(V_{2n} - C_{b_1}) \cup \{\bar{0}\}$. Define a block code of length n over $L = \{0, 1\}^4$ as follows:

$$K(C_{b_1}, f, C_{b_2}) \triangleq \{\varphi(\bar{u} \oplus f(\bar{v})) * \varphi(\bar{v}) : \bar{u} \in C_{b_1} \text{ and } \bar{v} \in C_{b_2}\},$$

where the mapping φ is defined by (5.8). Clearly

$$|K(C_{b_1}, f, C_{b_2})| = 2^{k_1 + k_2}.$$

It can be shown that

$$\begin{aligned} \underline{D}[K(C_{b_1}, f, C_{b_2})] &= \min\{\delta_1 d_1, \delta_{2,0} d_3, \underline{D}[C']\} \\ &\geq \min\{\delta_{2,0} d_3, \delta_H[C_{b_1} \oplus f(C_{b_2})] d_1\}, \end{aligned}$$

where $\delta_{2,0}$ denotes the minimum Hamming distance of $C_{b_{2,0}} (\triangleq \{\bar{v} \in C_{b_2} : f(\bar{v}) = \bar{0}\})$ and $C' \triangleq \{\varphi(\bar{u} \oplus f(\bar{v})) * \varphi(\bar{v}) : \bar{u} \in C_{b_1} \text{ and } \bar{v} \in C_{b_2} - C_{b_{2,0}}\}$.

6. Performance Analysis

In this section we assume that the signal set S satisfies condition (S1) and that every codeword of C is equally likely to be transmitted. Let p_{ic} be the probability of an incorrect decoding for a block. For a codeword \bar{u} in C , let $P_{ic}(\bar{u})$ be the probability of an incorrect decoding when \bar{u} is transmitted.

For a block code C over S of length n and an n -tuple \bar{u} over S , let $C[\bar{u}]$ be defined as

$$C[\bar{u}] \triangleq \{\bar{v} - \bar{u} : \bar{v} \in C\}, \quad (6.1)$$

where $-$ denotes the component-wise subtraction. We shall only consider decoding of a code with the following property:

(De) For two codewords \bar{u} and \bar{v} in C , if there is a one-to-one correspondence between $C[\bar{u}]$ and $C[\bar{v}]$ which preserves the distance measure d between two n -tuples, then

$$P_{ic}(\bar{u}) = P_{ic}(\bar{v}). \quad (6.2)$$

Most of the proposed decoding algorithms have the above property. If $\bar{S} = S$ (i.e., S is an abelian additive group under addition $+$ (e.g., $S_{2^L, \text{PSK}}$)) and C is linear with respect to $+$, then for any \bar{u} in C , $C[\bar{u}] = C$ and therefore

$$p_{ic} = P_{ic}(\bar{0}), \quad (6.3)$$

where $\bar{0}$ denotes the all zero n -tuple. For a linear code C with respect to \oplus , the above equality is not necessarily true. For a zero-tail Ungerboeck's 8-PSK TCM code [1] of length 9 with the minimum squared Euclidean distance $d_1 + 2d_2$ where d_1 and d_2 are defined by (2.15), the number of codewords with the minimum squared Euclidean distance from codeword $\bar{u} = (4, 1, 2, 0, \dots, 0)$ is 12, whereas that from the all-zero codeword $\bar{0}$ is 13, and simulation result on $P_{ic}(\bar{0})$ and $P_{ic}(\bar{u})$ by the maximum likelihood decoding for an additive white Gaussian noise (AWGN) channel at SNR per information bit = 6 dB shows a difference beyond the upper limit of confidence interval [34].

For a block code C over S (in general, a nonlinear code with respect to $+$), let C_0 be defined as

$$C_0 \triangleq \{\bar{u} \mid C[\bar{u}] = C \text{ and } \bar{u} \in C\}.$$

Then C_0 is not empty if and only if $\bar{0} \in C$. For $S_{2^L, \text{PSK}}$, any codeword in a linear code C with respect to \oplus whose components are 0 or 2^{L-1} is in C_0 .

Lemma 4 is useful for performance analysis of 2^L -PSK code.

Lemma 4: Suppose that $\bar{0} \in C$. Then the following properties hold:

- (1) C_0 is closed under the component wise $+$ addition.
- (2) For codewords \bar{u} and \bar{v} in C , $C[\bar{u}] = C[\bar{v}]$ if and only if $\bar{u} - \bar{v} \in C_0$.
- (3) Suppose that a subset C'_0 of C is closed under the component wise $+$ addition. Then $C'_0 \subseteq C_0$ if and only if there is a subset of C'_1 of C such that

$$\{\bar{v} + \bar{u} : \bar{v} \in C'_0 \text{ and } \bar{u} \in C'_1\} = C. \quad (6.4)$$

Proof: See Appendix D. $\Delta\Delta$

Suppose that C'_0 and C'_1 meet the conditions stated in (3) of the above lemma. Let $\bar{u}_1 (= \bar{0}), \bar{u}_2, \dots, \bar{u}_q$ be representatives of cosets of C'_0 in C , where $q = |C|/|C'_0|$. Then it follows from (De) and (2) of Lemma 4 that for \bar{v} in C'_0 and $1 \leq j \leq q$,

$$C[\bar{v} + \bar{u}_j] = C[\bar{u}_j].$$

Hence we have that

$$p_{ic} = \frac{1}{q} \sum_{j=1}^q P_{ic}(\bar{u}_j). \quad (6.5)$$

Example 6.1: Consider the basic 8-PSK code $C \triangleq P_{16}^\perp * RM_{4,2} * P_{16}$ with $R[C] = 27/32$ and $D[C] = 8 [26, 34]$. Let C'_0 and C'_1 be defined as

$$\begin{aligned} C'_0 &\triangleq P_{16}^\perp * RM_{4,1} * P_{16}, \\ C'_1 &\triangleq \{\bar{0} * \bar{u} * \bar{0} : \bar{u} \in (RM_{4,2} - RM_{4,1}) \cup \{\bar{0}\}\}. \end{aligned}$$

Then Lemma 2 in [35] (or Lemma A in [34]) implies that C'_0 is closed under the component-wise modulo-8 addition (denoted $+_8$) and

$$\{\bar{v} +_8 \bar{u} : \bar{v} \in C'_0 \text{ and } \bar{u} \in C'_1\} = C.$$

Let $\bar{u}_2^{(2)}$ and $\bar{u}_3^{(2)}$ be codewords in $RM_{4,2}$ represented by boolean polynomials x_1x_2 and $x_1x_2 \oplus x_3x_4$ respectively. Note that C , C'_0 and $RM_{4,2}$ are invariant under a permutation (called a linear transformation) on the bit positions induced by an invertible linear transformation on the binary representations of bit positions numbered 0 to 15. $RM_{4,2}$ consists of 64 cosets of $RM_{4,1}$ which are partitioned into three equivalent classes under linear transformations [31, Ch.15.2]. The first class is $RM_{4,1}$ itself. The second class consists of 35 cosets whose representatives are obtained from $\bar{u}_2^{(2)}$ by linear transformations, and the third class consists of 28 cosets whose representatives are obtained from $\bar{u}_3^{(2)}$ by linear transformations. Consequently it follows from (De) and (6.5) that

$$p_{ic} = \frac{1}{64} \{P_{ic}(\bar{u}_1) + 35P_{ic}(\bar{u}_2) + 28P_{ic}(\bar{u}_3)\},$$

where $\bar{u}_1 \triangleq \bar{0} * \bar{0} * \bar{0}$, $\bar{u}_2 \triangleq \bar{0} * \bar{u}_2^{(2)} * \bar{0}$ and $\bar{u}_3 \triangleq \bar{0} * \bar{u}_3^{(2)} * \bar{0}$. $C[\bar{u}_1]$, $C[\bar{u}_2]$ and $C[\bar{u}_3]$ have the same number, 1240, of nearest neighbor codewords from $\bar{0}$, but the numbers of the second nearest

neighbor codewords from $\bar{0}$ in $C[\bar{u}_1]$, $C[\bar{u}_2]$ and $C[\bar{u}_3]$ are 2048, 1024 and 1024, respectively. Simulation results on $P_{ic}(\bar{u}_1)$, $P_{ic}(\bar{u}_2)$ and $P_{ic}(\bar{u}_3)$ by the maximum likelihood decoding for an AWGN channel at SNR per information bit = 4.7dB are 4.41×10^{-3} , 4.11×10^{-3} and 4.05×10^{-3} , respectively. $\Delta\Delta$

Now we consider the maximum likelihood decoding for an AWGN channel. Let R^r denote the set of all r -tuples of real numbers. For $\bar{v} = (s_1, s_2, \dots, s_n)$ over \bar{S} , let $\sigma(\bar{v})$ denote the $2n$ -tuple in R^{2n} , represented by \bar{v} , and assume that for \bar{u} and $\bar{v} \in \bar{S}^n$, $d(\bar{u}, \bar{v})$ is defined as the squared Euclidean distance between $\sigma(\bar{u})$ and $\sigma(\bar{v})$. For $2n$ -tuples \bar{z} and \bar{z}' in R^{2n} , let (\bar{z}, \bar{z}') denote the inner product of \bar{z} and \bar{z}' . It follows from (2.3) and (2.22) that for \bar{u} and \bar{v} over \bar{S} ,

$$\begin{aligned} d(\bar{u}, \bar{v}) &= d(\bar{v} - \bar{u}, \bar{0}) \\ &= (\sigma(\bar{v}) - \sigma(\bar{u}), \sigma(\bar{v}) - \sigma(\bar{u})). \end{aligned} \quad (6.6)$$

We write $|\bar{u} - \bar{v}|_d$ for $d(\bar{u} - \bar{v}, \bar{0})$. Suppose that a codeword \bar{u} in C is transmitted and $\bar{z} \in R^{2n}$ is received. Since the probability of an incorrect decoding only depends on the squared Euclidean distances among $\sigma(\bar{u})$, \bar{z} and $\sigma(\bar{v})$ for all codewords \bar{v} other than \bar{u} , we consider $C[\bar{u}]$ instead of C and suppose that the all-zero n -tuple $\bar{0}$ in $C[\bar{u}]$ is transmitted. Decoding is correct if and only if

$$(\bar{z} - \sigma(\bar{v}), \bar{z} - \sigma(\bar{v})) > (\bar{z} - \sigma(\bar{0}), \bar{z} - \sigma(\bar{0})) \quad (6.7)$$

for every n -tuple \bar{v} in $C[\bar{u}]$ other than $\bar{0}$. The above inequality can be rewritten into the following inequality:

$$2(\sigma(\bar{v}) - \sigma(\bar{0}), \bar{z} - \sigma(\bar{0})) < |\bar{v}|_d. \quad (6.8)$$

For an n -tuple $\bar{v} \in C[\bar{u}]$, let $U(\bar{v})$ be the set of $2n$ -tuples over R, \bar{z} , which satisfy the inequality (6.7) (or (6.8)) and the probability, denoted $q_e(\bar{v})$, that a received $2n$ -tuple \bar{z} is not in $U(\bar{v})$ is given by [36]

$$q_e(\bar{v}) = Q\left(\sqrt{\frac{|\bar{v}|_d \rho}{2}}\right), \quad (6.9)$$

where $\rho = 2R[C]E_b/N_0$, E_b/N_0 denotes SNR per information bits and

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt.$$

A subset T of $C[\bar{u}] - \{\bar{0}\}$ is said to be $C[\bar{u}]$ -representative, if

$$\bigcap_{\bar{v} \in T} U(\bar{v}) = \bigcap_{\bar{v} \in C[\bar{u}] - \{\bar{0}\}} U(\bar{v}). \quad (6.10)$$

Since decoding is correct if and only if a received $2n$ -tuple \bar{z} is in the set of the right-hand side of (6.10), we have the following upper bound on $P_{ic}(\bar{u})$:

$$P_{ic}(\bar{u}) \leq \sum_{\bar{v} \in T} q_e(\bar{v}), \quad (6.11)$$

where T is a $C[\bar{u}]$ -representative set. For a distance measure g , a finite subset T of \bar{S}^n and a positive real number δ , let $N_{g,T}(\delta)$ denote the number of n -tuples \bar{v} in T such that $|\bar{v}|_g (\triangleq g(\bar{v}, \bar{0})) = \delta$, and let $\Delta_{g,T}$ be the set of positive real number δ such that $N_{g,T}(\delta) \neq 0$. Then it follows from (6.9) and (6.11) that we have the following union bound on $P_{ic}(\bar{u})$:

$$P_{ic}(\bar{u}) \leq \sum_{\delta \in \Delta_{g,T}} N_{g,T}(\delta) Q\left(\sqrt{\frac{\delta \rho}{2}}\right), \quad (6.12)$$

where T is a $C[\bar{u}]$ -representative set. In particular, if C is linear with respect to $+$, then it follows from (6.3) and (6.12) that

$$p_{ic} \leq \sum_{\delta \in \Delta_{d,T}} N_{d,T}(\delta) Q\left(\sqrt{\frac{\delta \rho}{2}}\right), \quad (6.13)$$

where T is a C -representative set. Since a distance measure satisfies (2.17) and the function $Q(\cdot)$ is monotonically decreasing, we can replace d by any distance measure in the right-hand side of (6.12). If C is linear with respect to \oplus , then we have the following upper bound on p_{ic} :

$$p_{ic} \leq \sum_{\delta \in \Delta_{w,T}} N_{w,T}(\delta) Q\left(\sqrt{\frac{\delta \rho}{2}}\right), \quad (6.14)$$

where w denotes the Euclidean weight measure defined by (4.4) and T is either C or the union of $C[\bar{u}_j]$ -representative sets over all \bar{u}_j in (6.5). Lemma 5 can be used for choosing a representative set.

Lemma 5: For \bar{v} , \bar{v}' and \bar{v}'' in \bar{S}^n , suppose that there are two nonnegative real numbers γ_1 , γ_2 such that

$$\sigma(\bar{v}'') - \sigma(\bar{0}) = \gamma_1(\sigma(\bar{v}) - \sigma(\bar{0})) + \gamma_2(\sigma(\bar{v}') - \sigma(\bar{0})), \quad (6.15)$$

$$|\bar{v}''|_d \geq \gamma_1 |\bar{v}|_d + \gamma_2 |\bar{v}'|_d. \quad (6.16)$$

Then it holds that

$$U(\bar{v}) \cap U(\bar{v}') \subseteq U(\bar{v}''). \quad (6.17)$$

Proof: Inequality (6.17) follows from the definition of $U(\bar{v})$. $\Delta\Delta$

If we choose \bar{v} and \bar{v}' as members of a representative set, then Lemma 5 shows that we don't need to choose \bar{v}'' as a member. If \bar{S} is a 2-dimensional lattice, this lemma can be easily applied.

Example 6.2: Consider the basic 4-level code, $P_8^\perp * RM_{3,1} * P_8 * V_8 (= \varphi[RM_{4,1}] * \varphi[P_{16}])$ over $S_{16-QASK}$, and assume that the squared signal configuration is used and the signal point labeled with 0000 is nearest to the center of the 2-dimensional signal set. Then the following upper bound on $P_{ic}(\bar{0})$ is derived.

$$P_{ic}(\bar{0}) \leq 3960Q \left(4\sqrt{\frac{2\rho}{5}} \right) + 14336Q \left(4\sqrt{\frac{3\rho}{5}} \right). \quad (6.18)$$

△△

Even for S_{2^t-PSK} , the above lemma is useful. Let s_j, s'_j and s''_j denote the j -th components of \bar{v}, \bar{v}' and \bar{v}'' over S_{2^t-PSK} respectively. There are two simple cases for which the above conditions (6.15) and (6.16) hold [22],

(i) For each j , either $s_j \cdot s'_j = 0$ or $s'_j = 2^{t-1} + s_j \pmod{2^t}$:

Let $\gamma_1 = \gamma_2 = 1$. If $s_j \cdot s'_j = 0$, then $s''_j = s_j + s'_j$, and otherwise, $s''_j = 2^{t-1}$.

(ii) Each component of \bar{v} is 0 or 2^{t-1} , and there is an $s \in S_{2^t-PSK}$ such that $0 \leq s \leq 2^{t-2}$ and s'_j is s or $2^t - s$ for every j with $s_j \neq 0$:

Let $\gamma_1 = \cos(2^{1-t}s\pi)$ and $\gamma_2 = 1$. If $s_j = 2^{t-1}$ and $s'_j = s$ (or $2^t - s$), then $s''_j = 2^{t-1} - s$ (or $2^{t-1} + s$), and otherwise, $s''_j = s'_j$.

By using these results, relatively small representative sets can be chosen to improve the upper bound (6.13) or (6.14) for several 8-PSK or 16-PSK codes [29, 35]. As examples, the following upper bounds on p_{ic} for $C_B^{(1)}$ and $C^{(1)}$ defined in part (4) of Example 5.1 are derived :

$$\begin{aligned} p_{ic} \leq & 120Q \left(\sqrt{2\rho} \right) + 128Q \left(\sqrt{4(2 - \sqrt{2})\rho} \right) \\ & + 1024Q \left(\sqrt{(8 - 3\sqrt{2})\rho} \right), \quad \text{for } C_B^{(1)}, \end{aligned} \quad (6.19)$$

$$\begin{aligned} p_{ic} \leq & 56Q \left(\sqrt{2\rho} \right) + 128Q \left(\sqrt{(5 - 2\sqrt{2})\rho} \right) \\ & + 64Q \left(\sqrt{4(2 - \sqrt{2})\rho} \right) + 256Q \left(\sqrt{(5 - \sqrt{2})\rho} \right) \\ & + 256Q \left(\sqrt{(8 - 3\sqrt{2})\rho} \right) + 256Q \left(\sqrt{2(4 - \sqrt{2})\rho} \right), \quad \text{for } C^{(1)}. \end{aligned} \quad (6.20)$$

△△

Let \bar{P}_{ic} denote an upper bound on p_{ic} , e.g., the right-hand sides of the above expressions. Let $P_{ic,s}$ denote the simulation result on p_{ic} . We use both \bar{P}_{ic} and $P_{ic,s}$ as measures of error performance of a block modulation code. Figures 2, 3 and 4 give the error performances of 8-PSK modulation codes $C^{(1)}$, $C^{(2)}$ defined in part (4) of Example 5.1, and 16-PSK modulation code $P_{32}^\perp * RM_{5,2} * P_{32} * V_{32}$ which is linear with respect to modulo-16 addition [35] respectively. These error performances are compared with those of uncoded QPSK systems for transmitting the same number of information bits. From Figures 2 and 3, we see that the difference between \bar{P}_{ic} and $P_{ic,s}$ is small for SNR greater than 6 dB per information bit. Table 2 (or 3) compares the error performances of $C^{(1)}$ and $C_B^{(1)}$ (or $C^{(2)}$ and $C_B^{(2)}$), and Table 4 compares those of $C^{(1)}$ and the zero-tail 4-state Ungerboeck's TCM code of length 9 given in Example 5.2. Table 5 compares those of 16-PSK codes, $P_{32}^\perp * \gamma(RM_{5,2}, P_{32}) * V_{32}$ and $P_{32}^\perp * RM_{5,2} * P_{32} * V_{32}$, both of which have rate 5/4, minimum squared Euclidean distance 4 and trellis diagrams of states 2^8 .

Figure 5 compares the error performances of the 16-QASK code given in Example 6.2 with that of an uncoded 8-AMPM. The error performance of this 16-QASK code is measured by two bit-error probabilities, \bar{p}_e and $p_{e,s}$ as follows:

$$\bar{p}_e \triangleq 1 - (1 - \bar{P}_{ic}(\bar{0}))^{1/k}, \quad (6.21)$$

$$p_{e,s} \triangleq 1 - (1 - P_{ic,s}(\bar{0}))^{1/k}, \quad (6.22)$$

where $\bar{P}_{ic}(\bar{0})$ is the value of the right-hand side of (6.18) and $P_{ic,s}(\bar{0})$ denotes simulation result on $P_{ic}(\bar{0})$. The error performance of the uncoded 8-AMPM is given by the bit-error probability, $p_{e,s}$ (simulation result).

7. Conclusion

In this paper, we have investigated the powerful multi-level technique for constructing bandwidth efficient block modulation codes. A general formulation for a multi-level block modulation code in terms of its component codes over substrings of labeling symbols has been presented. Lower bounds on the minimum distance of multi-level block modulation codes have been derived. Several specific methods for constructing component codes of a multi-level block modulation codes have been proposed. These methods provide interdependency between consecutive labelings of component codes. As a result, there is an inter-relationship

among the component codes of a multi-level block modulation code. This is a contrast to the construction of a basic multi-level block modulation code in which there is no interdependency among the component codes. A multi-level block modulation code with proper interdependency among its component codes has better error performance than its corresponding basic multi-level code.

We have also studied the linear structure of multi-level block modulation codes. Detail weight distribution of a linear multi-level modulation code and its enumeration have been discussed. Finally, error performance of block modulation codes has been analyzed for an AWGN channel based on a soft-decision maximum likelihood decoding. Error probabilities of some multi-level block modulation codes have been evaluated based on their Euclidean weight distributions and simulation results.

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Appendix A

Proof of Lemma 1:

Let $\bar{v} = (v_1, v_2, \dots, v_n)$ and $\bar{v}' = (v'_1, v'_2, \dots, v'_n)$ be two different codewords in C . Then we can express \bar{v} and \bar{v}' as the following forms:

$$\begin{aligned}\bar{v} &= \bar{v}^{(1)} * \bar{v}^{(2)} * \dots * \bar{v}^{(m)}, \\ \bar{v}' &= \bar{v}'^{(1)} * \bar{v}'^{(2)} * \dots * \bar{v}'^{(m)},\end{aligned}$$

where $\bar{v}^{(i)}$ and $\bar{v}'^{(i)}$ are codewords in C_i and

$$\begin{aligned}\bar{v}^{(i)} &= (v_{1i}, v_{2i}, \dots, v_{ni}), & v_{ji} &\in \bar{L}^{(i)}, \\ \bar{v}'^{(i)} &= (v'_{1i}, v'_{2i}, \dots, v'_{ni}), & v'_{ji} &\in \bar{L}^{(i)}, \\ v_j &= v_{j1}v_{j2} \cdots v_{jm}, \\ v'_j &= v'_{j1}v'_{j2} \cdots v'_{jm},\end{aligned}$$

with $1 \leq i \leq m$ and $1 \leq j \leq n$. Let h denote the first suffix such that

$$\bar{v}^{(h)} \neq \bar{v}'^{(h)}.$$

Since $v_{ji} = v'_{ji}$ for $1 \leq i < h$ and $1 \leq j \leq n$, it follows from (3.6) that

$$g_L(v_j, v'_j) = g_L(v_{j1}v_{j2} \cdots v_{jm}, v'_{j1}v'_{j2} \cdots v'_{jm}) \geq g_L^{(h)}(v_{jh}, v'_{jh}).$$

From (2.22) and (2.23) we have that

$$g_L(\bar{v}, \bar{v}') \geq g_L^{(h)}(\bar{v}^{(h)}, \bar{v}'^{(h)}) \geq D[g_L^{(h)}, C_h].$$

For $g_L = \underline{d}_L$, consider the case where $\bar{v}^{(i)} = \bar{v}'^{(i)}$ for $i \neq h$. It follows from (2.18), (2.19) and (3.6) that $\underline{d}_L(\bar{v}, \bar{v}') = \underline{d}_L^{(h)}(\bar{v}^{(h)}, \bar{v}'^{(h)})$. Hence $D[\underline{d}_L^{(h)}, C_h] \geq D[\underline{d}_L, C]$.

△△

Proof of Lemma 2:

- (1) It follows from (2.3) and (2.11) that for $a_1 \cdots a_{j-1} a_j a_{j+1} \cdots a_\ell$ and $a_1 \cdots a_{j-1} a'_j a'_{j+1} \cdots a'_\ell$ in L

$$\begin{aligned} d_L(a_1 \cdots a_{j-1} a_j a_{j+1} \cdots a_\ell, a_1 \cdots a_{j-1} a'_j a'_{j+1} \cdots a'_\ell) \\ = d(b_1 + \cdots + b_{j-1} + b_j + b_{j+1} + \cdots + b_\ell, b_1 + \cdots + b_{j-1} + b'_j + b'_{j+1} + \cdots + b'_\ell) \\ = d(b_j + b_{j+1} + \cdots + b_\ell, b'_j + b'_{j+1} + \cdots + b'_\ell), \end{aligned} \quad (\text{A.1})$$

where $b_h = \lambda_h(a_h)$ for $1 \leq h \leq \ell$ and $b'_h = \lambda_h(a'_h)$ for $j \leq h \leq \ell$. Equation (3.12) follows from (2.10), (3.6) and (A.1), and equation (3.13) follows from (3.6) and (A.1).

- (2) Let $\bar{v}^{(j)}$ be a codewords in C_j for $1 \leq j \leq m$ and $\bar{v}^{(i)}$ be a codeword in C_i different from $\bar{v}^{(i)}$. Then it follows from (2.22), (3.12), (3.13) and (A.1) that

$$\begin{aligned} d_L(\bar{v}^{(1)} * \cdots * \bar{v}^{(i-1)} * \bar{v}^{(i)} * \bar{v}^{(i+1)} * \cdots * \bar{v}^{(m)}, \\ \bar{v}^{(1)} * \cdots * \bar{v}^{(i-1)} * \bar{v}^{(i)} * \bar{v}^{(i+1)} * \cdots * \bar{v}^{(m)}) \\ = d_L^{(i)}(\bar{v}^{(i)}, \bar{v}^{(i)}). \end{aligned} \quad (\text{A.2})$$

Equations (2.23) and (A.2) imply that $D[d_L^{(i)}, C_i] \geq D[C]$ for $1 \leq i < m$. Then (3.14) follows from Lemma 1. Similarly, (3.15) is shown.

- (3) Since $|B_i| = |L_i| = 2$, it follows from (2.12) and (3.12) that for $a \neq a'$ in L_i ,

$$d_L^{(i)}(a, a') = d_i. \quad (\text{A.3})$$

From (2.22), (2.23) and (A.3),

$$D[d_L^{(i)}, C_i] = \delta_i d_i. \quad (\text{A.4})$$

Equations (3.14) and (A.4) imply (3.16).

$\Delta\Delta$

Appendix B

Proof of Lemma 3

For $n/2$ -tuple $\bar{v} = (v_1, v_2, \dots, v_{n/2})$, let $f'(\bar{v})$ be defined by

$$f'(\bar{v}) \triangleq \sum_{i=1}^{k_{21}-k_{22}} (\bar{h}_i, \bar{v}) \bar{u}_{1i}. \quad (\text{B.1})$$

Then, for $n/2$ -tuples \bar{v} and \bar{v}' , $f(\bar{v}\bar{v}') = f'(\bar{v}') \circ f'(\bar{v})$. Let \bar{v} be a codeword of C_{21} . If and only if $\bar{v} \in C_{22}$, that is, \bar{v} is orthogonal to every codeword of $C_{22}^\perp - C_{21}^\perp$, then

$$f'(\bar{v}) = \bar{0}. \quad (\text{B.2})$$

Let C''_{11} denote the linear subcode of C_{11} generated by $\{\bar{u}_{1j} : 1 \leq j \leq k_{21} - k_{22}\}$. Then it follows from (B.2) that f' is a one-to-one mapping from C''_{21} onto C''_{11} .

Proof of (1)

For a nonzero codeword \bar{w} in $\gamma(C_1, C_2) (= F(C_{b1}, f, C_{b2}))$, \bar{w} can be expressed as

$$\bar{w} = \bar{w}_1 \circ \bar{w}_2,$$

$$\bar{w}_i = (\bar{u}_i \oplus \bar{u}'_i) * \bar{v}_i \quad \text{for } i = 1 \text{ and } 2,$$

where $\bar{u}'_i \in C'_{11}$, $\bar{v}_i \in C_{21}$, $\bar{u}_1 \triangleq f'(\bar{v}_2) \in C''_{11}$ and $\bar{u}_2 \triangleq f'(\bar{v}_1) \in C''_{11}$. Since $C'_{11} \cap C''_{11} = \{\bar{0}\}$, $\bar{u}_i \oplus \bar{u}'_i = \bar{0}$ if and only if $\bar{u}_i = \bar{u}'_i = \bar{0}$. If $\bar{u}_i \oplus \bar{u}'_i \neq \bar{0}$, then $|\bar{u}_i \oplus \bar{u}'_i|_H \geq \delta_1/2$, and if $\bar{v}_i \neq \bar{0}$, then $|\bar{v}_i|_H \geq \delta_2/2$. There are three cases to be considered.

(1) Suppose that $\bar{u}_i \oplus \bar{u}'_i \neq \bar{0}$ for $i = 1$ and 2 . Then,

$$|\bar{w}|_d \geq |\bar{u}_1 \oplus \bar{u}'_1|_H d_1 + |\bar{u}_2 \oplus \bar{u}'_2|_H d_1 \geq \delta_1 d_1.$$

(2) Suppose that $\bar{u}_1 \oplus \bar{u}'_1 = \bar{0}$ and $\bar{u}_2 \oplus \bar{u}'_2 \neq \bar{0}$. Then $\bar{u}_1 = \bar{u}'_1 = \bar{0}$, and $|\bar{u}_2 \oplus \bar{u}'_2|_H \geq \delta_1/2$.

Since $\bar{u}'_1 = \bar{0}$, $\bar{u}'_2 \in C_{12}$. If $\bar{u}_2 = \bar{0}$, then $|\bar{u}'_2|_H \geq \delta_1$ and $|\bar{w}|_d \geq \delta_1 d_1$. If $\bar{u}_2 \neq \bar{0}$, it follows from (B.1) that $\bar{v}_1 \neq \bar{0}$ and therefore $|\bar{v}_1|_H \geq \delta_2/2$. Hence $|\bar{w}|_d \geq \delta_2 d_2/2 + \delta_1 d_1/2$.

(3) Suppose that $\bar{u}_1 \oplus \bar{u}'_1 = \bar{u}_2 \oplus \bar{u}'_2 = \bar{0}$. Then $\bar{u}_1 = \bar{u}'_1 = \bar{u}_2 = \bar{u}'_2 = \bar{0}$. If $\bar{v}_1 \neq \bar{0}$ and $\bar{v}_2 \neq \bar{0}$, then $|\bar{w}|_d \geq (|\bar{v}_1|_H + |\bar{v}_2|_H) d_2 \geq \delta_2 d_2$. If $\bar{v}_2 = \bar{0}$, then $\bar{v}_1 \neq \bar{0}$. It follows from (B.2) that $\bar{v}_1 \in C_{22}$ and $|\bar{v}_1|_H \geq \delta_2$. Hence $|\bar{w}|_d \geq \delta_2 d_2$. Thus inequality (5.32) holds.

Let \bar{v}_1 be a codeword of Hamming weight δ_2 in C_{22} . Then $f(\bar{v}_1 \circ \bar{0}) = \bar{0} \circ \bar{0}$ and $|f(\bar{v}_1 \circ \bar{0}) * (\bar{v}_1 \circ \bar{0})|_d = |\bar{v}_1|_H d_2 = \delta_2 d_2$.

△△

The proof above implies (2) of the lemma.

Proof of (3)

For a linear block code C of length n with respect to \oplus and a positive integer t such that $1 < t < n$, let C_f (or C_p) denote the set of those codewords of C whose first t (or last $n - t$) components are all-zero. Let C and C' denote $C_1 * C_2$ and $\gamma(C_1, C_2)$ respectively, where $C_i = \mu(C_{i1}, C_{i2})$. We show that

$$|C'_p| \geq |C_p|, \quad (B.3)$$

$$|C'_f| \geq |C_f|. \quad (B.4)$$

Then the claim is proved [15, Appendix]. From the symmetry shown by (5.26), it is sufficient to consider t such that $1 < t \leq n/2$.

Let a codeword \bar{w} in C be expressed as

$$\bar{w} = (\bar{w}_{11} \circ (\bar{w}_{11} \oplus \bar{w}_{12})) * (\bar{w}_{21} \circ (\bar{w}_{21} \oplus \bar{w}_{22})) \quad (B.5)$$

where $\bar{w}_{ij} \in C_{ij}$ for i and j in $\{1, 2\}$.

(1) Suppose that $\bar{w}_{11} \oplus \bar{w}_{12} = \bar{w}_{21} \oplus \bar{w}_{22} = \bar{0}$. Then $\bar{w}_{11} \in C_{12}$ and $\bar{w}_{21} \in C_{22}$. Since $\bar{w}_{11} \circ \bar{0} \in C_{b1}$, $\bar{w}_{21} \circ \bar{0} \in C_{b2}$ and $f(\bar{w}_{21} \circ \bar{0}) = \bar{0} \circ \bar{0}$, \bar{w} is also in C' . That is, inequality (B.3) holds.

(2) Suppose that $\bar{w} \in C_f$. Then \bar{w}_{11} can be expressed as

$$\bar{w}_{11} = \bar{u}' \oplus \bar{u}'', \quad (B.6)$$

where $\bar{u}' \in C'_{11}$ and $\bar{u}'' \in C''_{11}$. Since f' is a one-to-one mapping from C''_{21} onto C'_{11} , there is \bar{v} in C_{21} such that $\bar{u}'' = f'(\bar{v})$. From (B.2),

$$f'(\bar{v} \oplus \bar{w}_{22}) = f'(\bar{v}) = \bar{u}''. \quad (B.7)$$

Since $\bar{u}' \circ (\bar{u}' \oplus \bar{w}_{12}) \in C_{b1}$ and $\bar{w}_{21} \circ (\bar{v} \oplus \bar{w}_{22}) \in C_{b2}$, it follows from (B.6) and (B.7) that $(\bar{w}_{11} \circ (\bar{u}' \oplus \bar{w}_{12} \oplus f(\bar{w}_{21}))) * (\bar{w}_{21} \circ (\bar{v} \oplus \bar{w}_{22}))$, denoted \bar{w}' , is in C'_f . Since different \bar{w}' results from different \bar{w} , inequality (B.4) holds.

△△

Appendix C

A Trellis Diagram for the Second Code Given in Table 1

For $1 \leq j \leq 8$, the j -th input symbol is $a_j b_j$, where a_j is the least significant bit. Let (A_j, B_j) be the state of the overall encoder after the j -th symbol, $a_j b_j$, is received. Assume that the initial state is $(A_0, B_0) = (0, 0)$. The state transition from (A_{j-1}, B_{j-1}) to (A_j, B_j) when j -th symbol $a_j b_j$ is received is defined as follows:

- (1) $A_1 = a_1, B_1 = b_1$.
- (2) For $2 \leq j \leq 4$, if $a_j = A_{j-1}$, then $A_j = A_{j-1}$ and $B_j = B_{j-1} \oplus b_j$, and otherwise, the state transition is not defined.
- (3) For $5 \leq j \leq 8$, if $a_j = B_{j-1}$, then $A_j = A_{j-1} \oplus b_j$ and $B_j = B_{j-1}$ and otherwise, the state transition is not defined.

An input sequence is accepted as a code sequence if and only if every state transition is defined and $A_8 = 0$.

Appendix D

Proof of Lemma 4:

- (D) Suppose that for an n -tuple \bar{u} over S , $C[\bar{u}] = C$. Since $\bar{0} \in C$, $-\bar{u} \in C$. Therefore, for any nonnegative integer h , $-h\bar{u} \in C$. Since C is finite, there is a positive integer h such that $-h\bar{u} = \bar{0}$, i.e., $\bar{u} = -(h-1)\bar{u} \in C$. Hence $\bar{u} \in C_0$.
- (1) Let \bar{v}_1 and \bar{v}_2 be codewords in C_0 . Then, for any codeword \bar{u} , there is a codeword \bar{u}' such that $\bar{u} - \bar{v}_1 = \bar{u}'$ and $\bar{u}' - \bar{v}_2 \in C$. Hence $\bar{u} - (\bar{v}_1 + \bar{v}_2) = \bar{u}' - \bar{v}_2 \in C$. That is, $C[\bar{v}_1 + \bar{v}_2] = C$ and from (D), $\bar{v}_1 + \bar{v}_2 \in C_0$.
- (2) Only if part: For any codeword \bar{u}' in C , there is a codeword \bar{v}' in C such that $\bar{u}' - \bar{u} = \bar{v}' - \bar{v}$. Hence $\bar{u}' - (\bar{u} - \bar{v}) \in C$. That is, $C[\bar{u} - \bar{v}] = C$ and from (D), $\bar{u} - \bar{v} \in C_0$.
If part: If $\bar{u} - \bar{v} \in C_0$, then for any \bar{u}' in C , there is a codeword \bar{v}' such that $\bar{u}' - (\bar{u} - \bar{v}) = \bar{v}'$, i.e., $\bar{u}' - \bar{u} = \bar{v}' - \bar{v}$. That is, $C[\bar{u}] \subseteq C[\bar{v}]$. Since $|C[\bar{u}]| = |C[\bar{v}]|$, $C[\bar{u}] = C[\bar{v}]$.
- (3) If part : For any \bar{u} in C , there exist \bar{v}' in C'_0 and \bar{u}' in C'_1 such that $\bar{u} = \bar{v}' + \bar{u}'$. Let \bar{v} be a codeword in C'_0 . Then we have that

$$\bar{u} - \bar{v} = \bar{v}' - \bar{v} + \bar{u}'. \quad (\text{D.1})$$

Since C'_0 is finite and closed under the component wise + addition, $-\bar{v} \in C'_0$ and therefore, $\bar{v}' - \bar{v} \in C'_0$. It follows from (6.4) and (D.1) that $\bar{u} - \bar{v} \in C$. That is, $C[\bar{v}] = C$ and therefore, $\bar{v} \in C_0$.

Only if part: For any \bar{v} in C'_0 , $-\bar{v}$ is also in C'_0 . Then for any \bar{u} in C , $\bar{u} - (-\bar{v}) = \bar{u} + \bar{v} \in C$. Then C can be taken as C'_1 .

△△

Table 1:
Some Comparisons of $\gamma(C_1, C_2)$ with $C_1 * C_2$

Definition of C	n	$\log_2 C $	$D[C]/d_2$	The number of states of a trellis diagram	$N_{dC}(D[C])$
$P_8^\perp * P_8$	8	8	2	4 (8 section)	28
$\gamma(P_8^\perp, P_8)$	8	8	2	4 (8 section)	12
$RM_{4,1} * P_{16}$	16	20	2	16 (4 section)	120
$\gamma(RM_{4,1}, P_{16})$	16	20	2	16 (4 section)	56
$RM_{5,2} * P_{32}$	32	47	2	128 (4 section)	496
$\gamma(RM_{5,2}, P_{32})$	32	47	2	128 (4 section)	240
$RM_{5,1} * RM_{5,3}$	32	32	4	256 (4 section)	1240
$\gamma(RM_{5,1}, RM_{5,3})$	32	32	4	256 (4 section)	280

Table 2:
Comparison of the Error Performances between the Codes,
 $C_B^{(1)}$ and $C^{(1)}$, Given in Example 5.1

SNR per information bit (dB)	$C_B^{(1)}$		$C^{(1)}$	
	Simulation result $P_{ic,s}$	Upper bound \bar{P}_{ic}	Simulation result $P_{ic,s}$	Upper bound \bar{P}_{ic}
6.0	4.17E-03	5.09E-03	3.60E-03	4.52E-03
8.0	3.27E-05	3.50E-05	2.60E-05	2.74E-05
10.0		1.66E-08		1.07E-08

Table 3:
Comparison of the Error Performances between the Codes,
 $C_B^{(2)}$ and $C^{(2)}$, Given in Example 5.1

SNR per information bit (dB)	$C_B^{(2)}$		$C^{(2)}$	
	Simulation result $P_{ic,s}$	Upper bound \bar{P}_{ic}	Simulation result $P_{ic,s}$	Upper bound \bar{P}_{ic}
6.08	8.05E-03	1.33E-02	7.38E-03	1.26E-02
8.08	2.94E-05	3.03E-05	2.41E-05	2.53E-05
10.08		3.68E-09		2.43E-09

Table 4:

Comparison of the Error Performances between $C^{(1)}$ and
the Zero-tail 4-state Ungerboeck's 8-PSK TCM Code of Length 9

SNR per information bit (dB)	$C^{(1)}$		The zero-tail 4-state Ungerboeck's code	
	Simulation result $P_{ic,s}$	Upper bound \bar{P}_{ic}	Simulation result $P_{ic,s}$	Upper bound \bar{P}_{ic}
8.0	2.60E-05	2.74E-05	1.76E-05	1.80E-05
10.0		1.07E-08		1.45E-08
12.0		5.88E-14		3.07E-13

Table 5:

Comparison of the Error Performances between 16-PSK Codes

$$P_{32}^{\perp} * RM_{5,2} * P_{32} * V_{32} \text{ and } P_{32}^{\perp} * \gamma(RM_{5,2}, P_{32}) * V_{32}$$

SNR per information bit (dB)	$P_{32}^{\perp} * RM_{5,2} * P_{32} * V_{32}$		$P_{32}^{\perp} * \gamma(RM_{5,2}, P_{32}) * V_{32}$	
	Simulation result $P_{ic,s}$	Upper bound \bar{P}_{ic}	Simulation result $P_{ic,s}(\bar{0})$	Upper bound $\bar{P}_{ic}(\bar{0})$
7.02	2.64E-03	4.67E-03	2.24E-03	4.33E-03
8.02		8.15E-05		6.43E-05
9.02		6.93E-07		4.95E-07
10.02		2.22E-09		1.45E-09

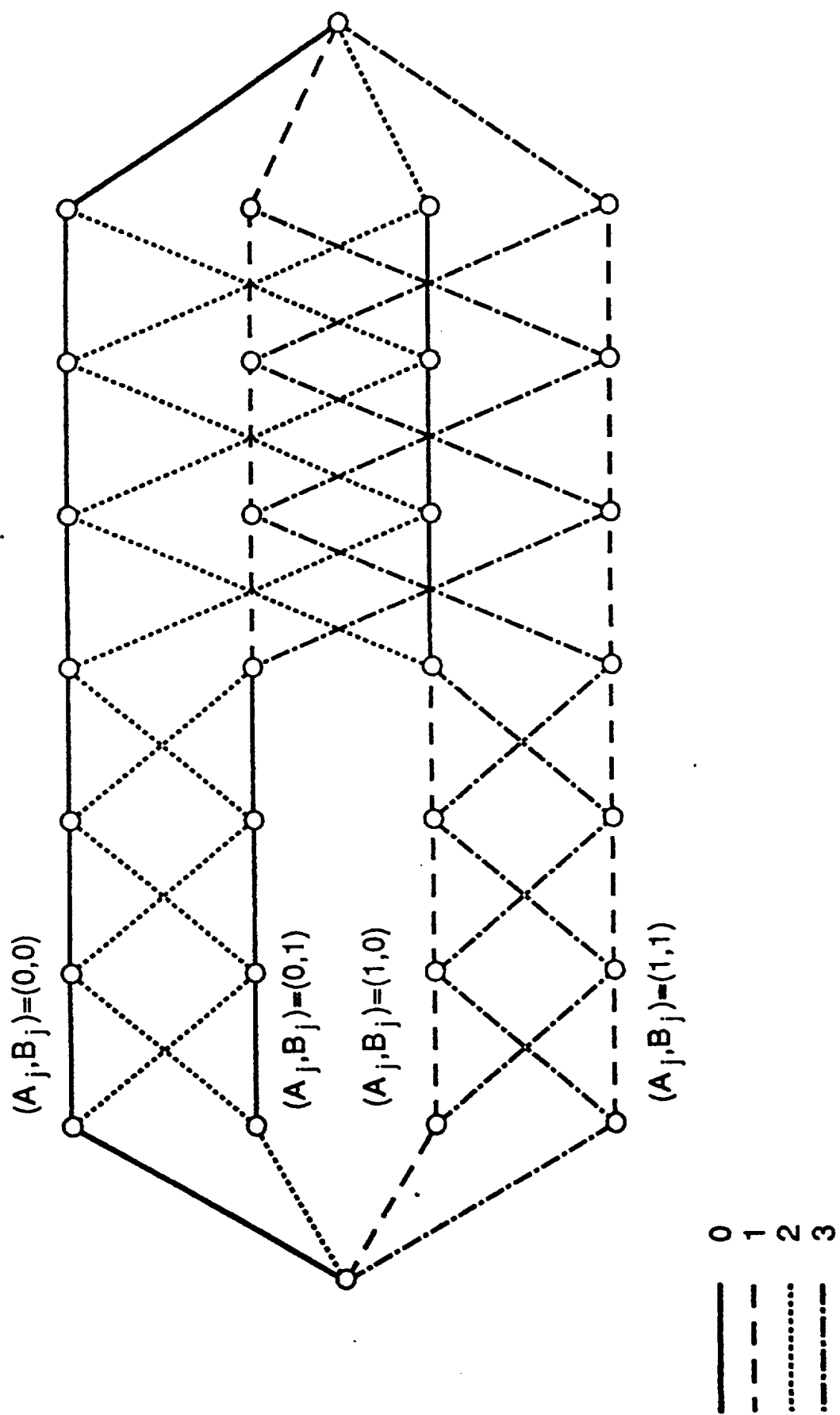


Fig. 1 A 4-state trellis diagram for the code defined in part (1) of Example 5.1

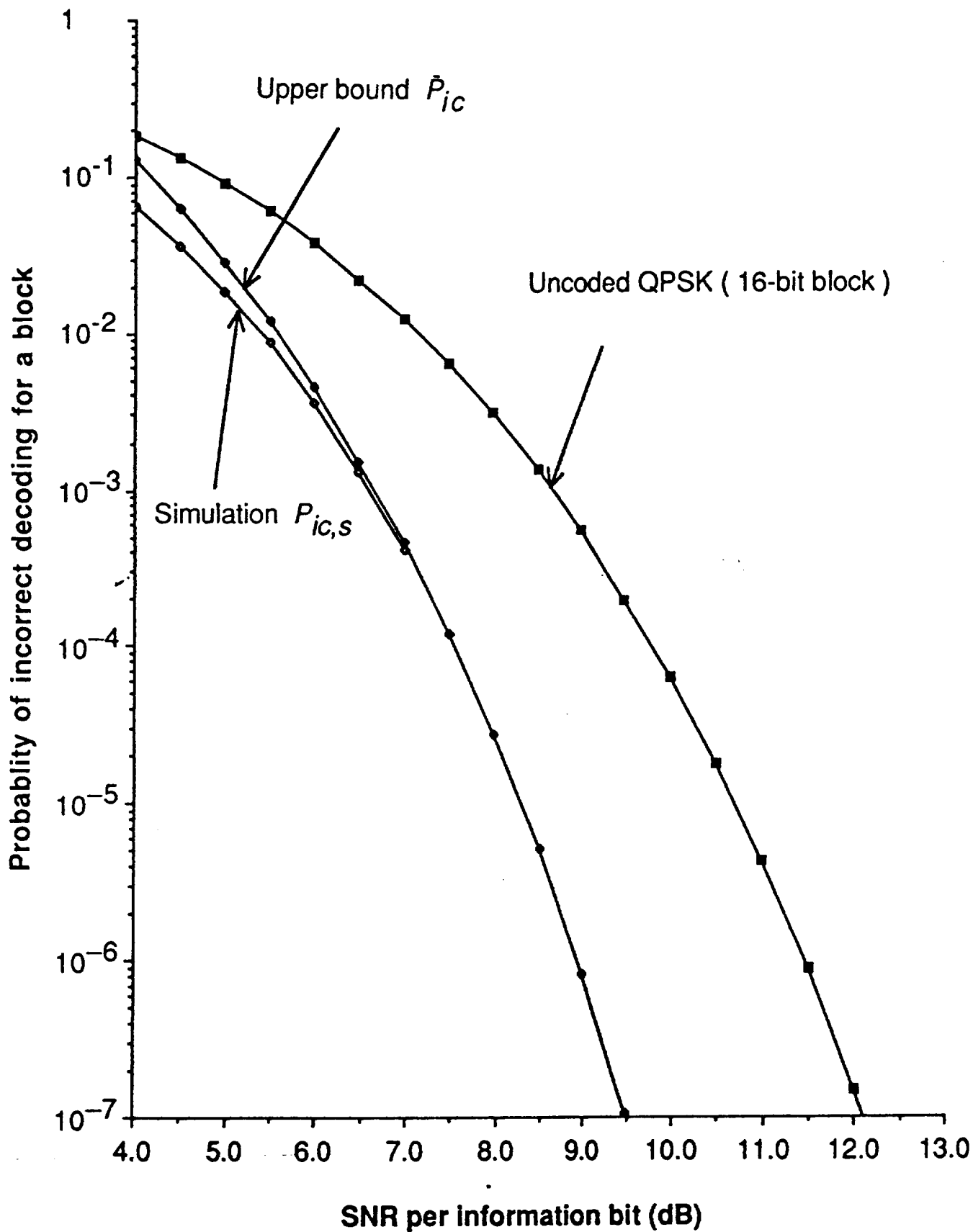


Fig. 2 Error performance of the 4-state 8-PSK code $C^{(1)}$ given in Example 5.1

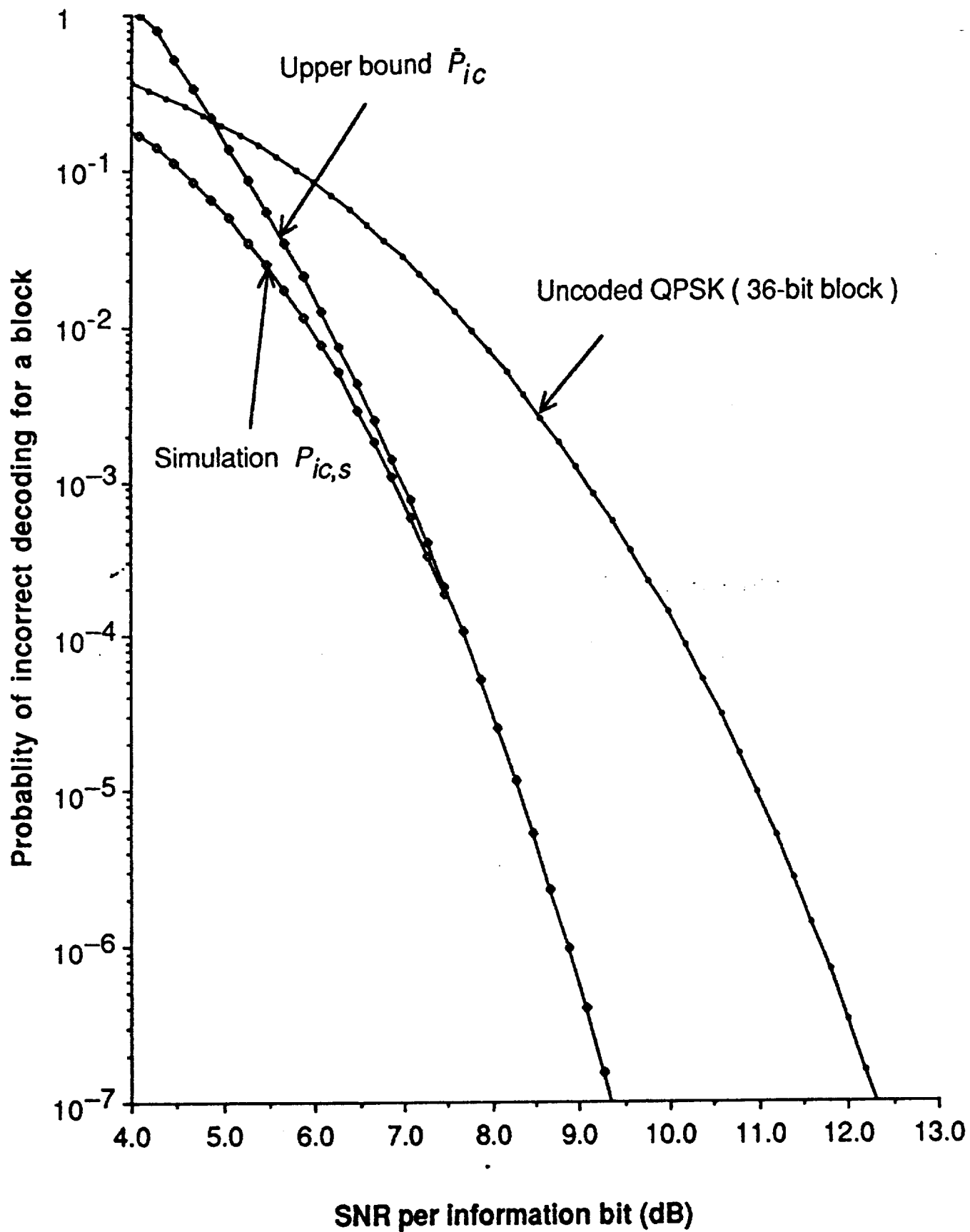


Fig. 3 Error performance of the 16-state 8-PSK code $C^{(2)}$ given in Example 5.1

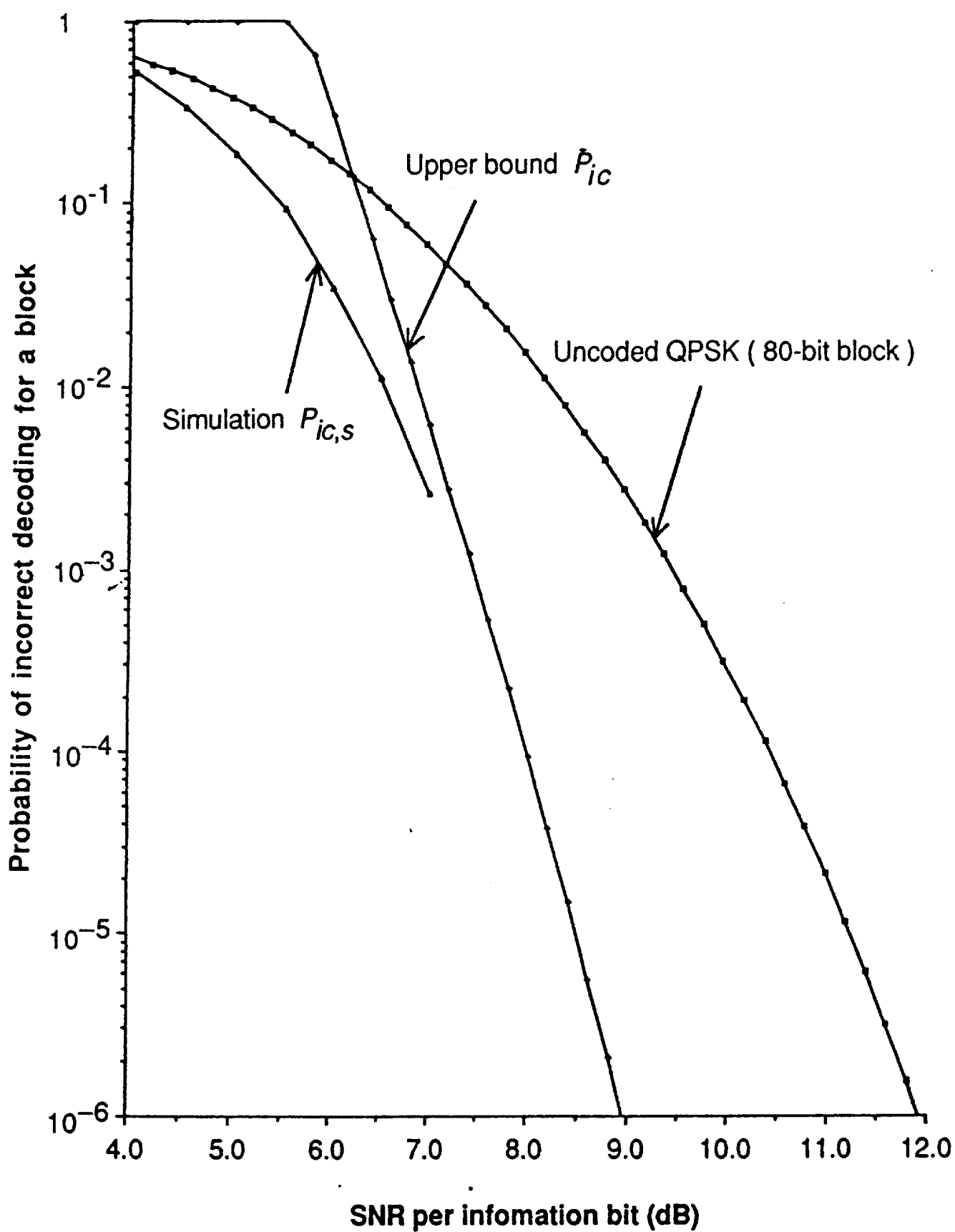


Fig. 4 Error performance of the basic 4-level 256-state 16-PSK code $P_{32}^{\perp} * RM_{3,2} * P_{32} * V_{32}$

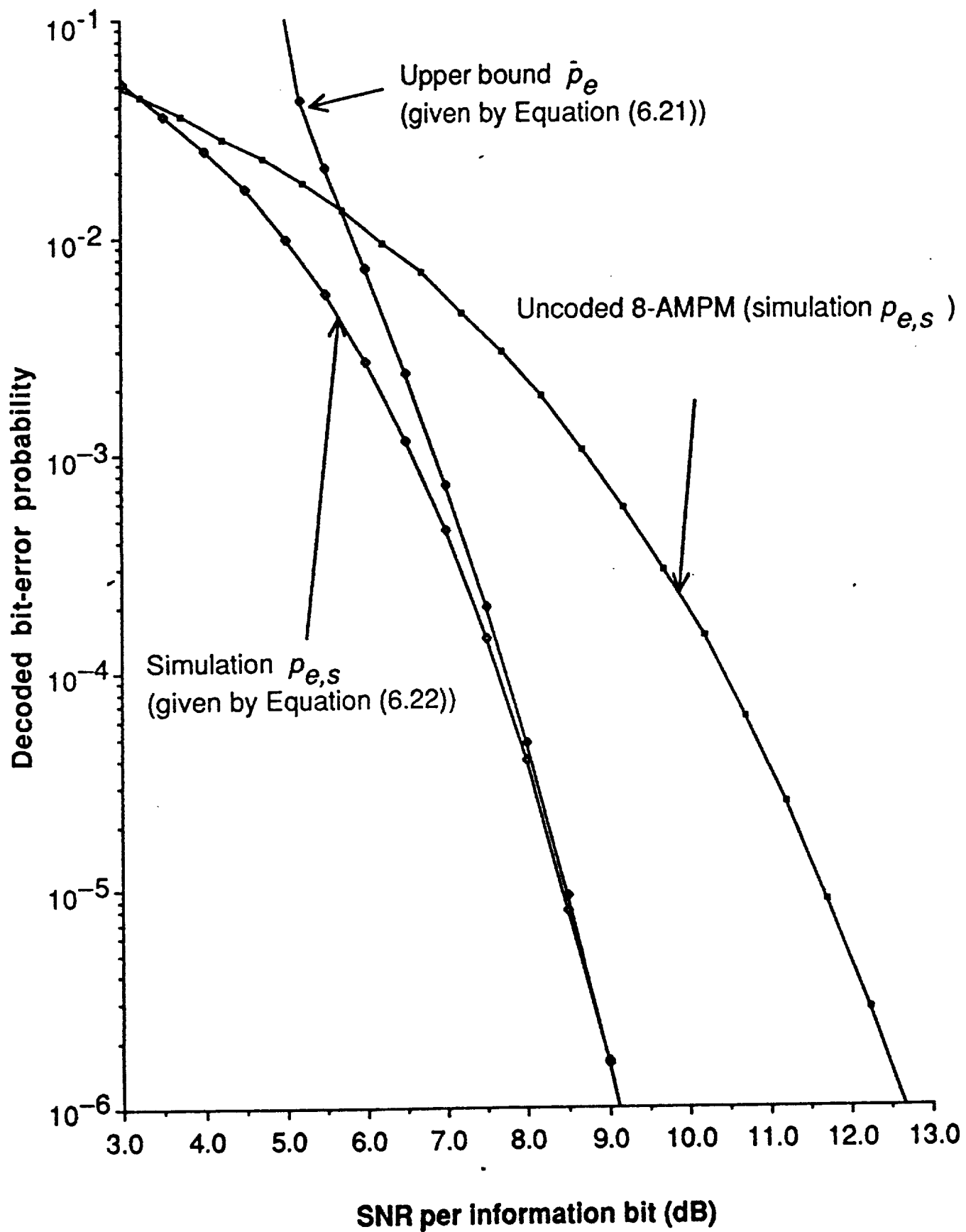


Fig. 5 Error performances of the 16-QASK code given in Example 6.2 and the uncoded 8-AMPM